COMPUTING THE $\sin_p$ FUNCTION VIA THE INVERSE POWER METHOD

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ABSTRACT. In this paper, we discuss a new iterative method for computing $\sin_p$. This function was introduced by Lindqvist in connection with the unidimensional nonlinear Dirichlet eigenvalue problem for the p-Laplacian. The iterative technique was inspired by the inverse power method in finite dimensional linear algebra and is competitive with other methods available in the literature.
Keywords: p-Laplacian, eigenvalues, eigenfunctions, $\sin_p$, inverse power method.

1 Introduction

In this paper we present a new method to compute the function $\sin_p$, inspired by recent work done by the authors in [BEM], where an iterative algorithm based on the inverse power method of linear algebra was introduced for the computation of the first eigenvalue and first eigenfunction of the Dirichlet problem for the p-Laplacian in arbitrary domains in $\mathbb{R}^N$. The functions $\sin_p$, $1 < p < \infty$, can be thought of as generalizations of the familiar trigonometric functions. They arise in the unidimensional Dirichlet eigenvalue problem for the p-Laplacian and were introduced in this capacity in [Lindqvist], where a power series formula for computing them was also formally given.

In [BR1] $\sin_p$ functions were utilized to introduce a generalization of the Prüfer transformation and thus represent, in two phase-plane coordinates, Sturm-Liouville-type problems involving the N-dimensional radially symmetric p-Laplacian $L_p u := x^{1-N}\left(x^{N-1}\left|u'\right|^{p-2}u'\right)'$, $0 \leqslant a < x < b < \infty$. This approach was numerically implemented in [BR2] for an eigenvalue problem involving L_p with separated homogeneous boundary conditions. In that paper an interpolation table for $\sin_p$ was obtained by numerically solving an ODE. Also in that paper the authors raised the question of finding a fast and accurate algorithm for computing $\sin_p$.

Our method depends on the convergence of a sequence of functions whose definition, as in [BEM], is motivated by an extension of the inverse power method of linear algebra for

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obtaining the first eigenvalue and first eigenfunction of finite dimensional linear operators. These functions are recursively defined and can be given in integral form, so that they can be obtained by numerical integration.

More specifically, recall that it suffices to obtain $\sin_p$ in the interval $I_p = [0, \pi_p/2]$, since it is extended to the interval $[\pi_p/2, \pi_p]$ symmetrically with respect to $\pi_p/2$ and afterward to the whole real line $\mathbb{R}$ as an odd, $2\pi_p$-periodic function (the definition of $\sin_p$ as well as the precise value of π_p are recalled in Section 2). We define the following sequence of (positive) functions $\{\phi_n\} \subset C^1(I_p)$. Set $\phi_0 \equiv 1$ and

$$\begin{cases} \left(\phi'_{n+1} \left|\phi'_{n+1}\right|^{p-2}\right)' = -\phi_n \left|\phi_n\right|^{p-2} & \text{if } x \in I_p, \\ \phi_{n+1}(0) = \phi'_{n+1}(\pi_p/2) = 0. \end{cases}$$

We prove that the scaled sequence $\left\{\sqrt[p]{p-1}\phi_n / \left\|\phi_n\right\|_\infty\right\}$ converges uniformly to $\sin_p$ in I_p. The functions ϕ_n can be written in integral form as

$$\phi_{n+1}(x) = \int_0^x \left(\int_\theta^{\pi_p/2} \phi_n(s)^{p-1}\, ds\right)^{\frac{1}{p-1}} d\theta, \quad x \in I_p,$$

and, therefore, are readily computed using standard efficient numerical methods for definite integrals.

This paper is organized as follows. In Section 2, we recall the definition and some basic properties of $\sin_p$ which will be used in the sequel. In Section 3, we show how to recursively construct a sequence of functions which converge uniformly to $\sin_p$. Finally, in Section 4 we compare the performance of our method with those of [Lindqvist] and [BR2].

2 The function $\sin_p$

For the sake of completeness we recall in this section the definition and some properties of the function $\sin_p$. The unidimensional Dirichlet eigenvalue problem for the p-Laplacian, $p > 1$, is

$$\begin{cases} \psi_p(u')' = -\lambda\psi_p(u) & \text{if } a < x < b, \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $\psi_p(t) = t\,|t|^{p-2}$.

It is easy to verify that if λ_1 is the first eigenvalue of

$$\begin{cases} \psi_p(v')' = -\lambda\psi_p(v) & \text{if } a < x < m := \dfrac{a+b}{2}, \\ v(a) = v'(m) = 0, \end{cases} \tag{2}$$

and v_1 is the corresponding positive eigenfunction, then λ_1 is also the first eigenvalue for (1) with

$$u_1(x) = \begin{cases} v_1(x) & \text{if } a \leqslant x \leqslant m, \\ v_1(a+b-x) & \text{if } m \leqslant x \leqslant b, \end{cases}$$

being the corresponding positive eigenfunction. Moreover, this function is stricly increasing on $[a, m)$, strictly decreasing on $(m, b]$ and has only one maximum point which is reached at $x = m$. Thus, $\|u_1\|_\infty = u_1(m)$.

An expression for λ_1 is well known and can be obtained by integration (see [Otani]) as follows. First multiply (1) by u_1' and integrate the resulting equation by parts on $[a, x]$ to obtain

$$\psi_p(u_1')\, u_1'\big|_a^x - \int_a^x \psi_p(u_1')\, u_1'' dx = -\lambda_1 \int_a^x \psi_p(u_1)\, u_1' dx. \tag{3}$$

We have

$$\psi_p(u_1')\, u_1'\big|_a^x = |u_1'(x)|^p - |u_1'(a)|^p \tag{4}$$

$$\int_a^x \psi_p(u_1)\, u_1' dx = \int_{u_1(a)}^{u_1(x)} \psi_p(s)\, ds = \frac{|u(x)|^p}{p} - \frac{|u(a)|^p}{p}, \tag{5}$$

$$\int_a^x \psi_p(u_1')\, u_1'' dx = \int_{u_1'(a)}^{u_1'(x)} \psi_p(s)\, ds = \frac{|u'(x)|^p}{p} - \frac{|u'(a)|^p}{p}. \tag{6}$$

Substituting (4), (5) and (6) in (3) we obtain

$$\left(1 - \frac{1}{p}\right)\left[|u_1'(x)|^p - |u_1'(a)|^p\right] = -\lambda_1 \left[\frac{|u_1(x)|^p}{p} - \frac{|u_1(a)|^p}{p}\right],$$

whence

$$\left[\left(1 - \frac{1}{p}\right)|u_1'|^p + \lambda_1 \frac{|u_1|^p}{p}\right]_a^x = 0.$$

This means that

$$\frac{p-1}{p}|u_1'|^p + \frac{\lambda_1}{p}|u_1|^p \equiv C,$$

where C is a constant and $p' = p/(p-1)$ is the conjugate of p. The value of C can be found computing the value of this expression at the maximum point m; choosing u_1 such that $u_1(m) = 1$ we find

$$C = \frac{p-1}{p}|u_1'(m)|^p + \frac{\lambda_1}{p}|u_1(m)|^p = \frac{\lambda_1}{p}.$$

Therefore,

$$(p-1)\,|u_1'(x)|^p + \lambda_1 |u_1(x)|^p = \lambda_1 \tag{7}$$

for all $x \in [a, b]$.

On the interval $[a, m]$ we have $u' \geqslant 0$, hence we can write

$$\frac{u_1'(x)}{\sqrt[p]{(1 - |u_1(x)|^p)}} = \sqrt[p]{\frac{\lambda_1}{p-1}} \tag{8}$$

for all $x \in [a, m]$. Integrating this equation on (a, m) leads to

$$\frac{b-a}{2}\sqrt[p]{\frac{\lambda_1}{p-1}} = \int_{u_1(a)}^{u_1(m)} \frac{ds}{\sqrt[p]{1-s^p}} = \int_0^1 \frac{ds}{\sqrt[p]{1-s^p}},$$

which gives the expression

$$\lambda_1 = (p-1)\left(\frac{2}{b-a}\int_0^1 \frac{ds}{\sqrt[p]{1-s^p}}\right)^p = \left(\frac{\pi_p}{b-a}\right)^p, \tag{9}$$

where we set

$$\pi_p := 2\sqrt[p]{p-1}\int_0^1 \frac{ds}{\sqrt[p]{1-s^p}}. \tag{10}$$

Making the change of variable $s = \sqrt[p]{t}$ in the last integral and using the classical Beta function B we obtain

$$\int_0^1 \frac{ds}{\sqrt[p]{1-s^p}} = \frac{1}{p}\int_0^1 t^{\frac{1}{p}-1}(1-t)^{-\frac{1}{p}}dt = \frac{1}{p}B\left(1-\frac{1}{p},\frac{1}{p}\right) = \frac{\pi/p}{\sin(\pi/p)}$$

(Here one use the properties $B(x,y)B(x+y,1-y) = x/x\sin(\pi y)$ and $B(1,z) = 1/z$ with $x = 1-1/p$ and $y = z = 1/p$).

Therefore,

$$\pi_p = \frac{2\sqrt[p]{p-1}\,(\pi/p)}{\sin(\pi/p)} \tag{11}$$

and

$$\lambda_1 = \left(\frac{2\sqrt[p]{p-1}\,(\pi/p)}{(b-a)\sin(\pi/p)}\right)^p.$$

When $a = 0$ and $b = \pi_p$ we denote the function $\sqrt[p]{p-1}u_1$ by $\sin_p$. Thus, $\sin_p(0) = 0 = \sin_p'(\pi_p/2)$, $\lambda_1 = 1$ and from (7):

$$\left|\sin_p'\right|^p + \frac{|\sin_p|^p}{p-1} = 1.$$

It is clear from this equation that $\sin_p'(0) = 1$.

We remark that $u = \sin_p$ is also the unique solution of the initial value problem

$$|u'|^p + \frac{|u|^p}{p-1} = 1, \quad u(0) = 0,$$

which can be used to define this function.

Alternatively, we can define $\sin_p$ on the interval $[0, \pi_p/2]$ as an inverse function. In fact, multiplying (8) by $\sqrt[p]{p-1}$ and using (9) with $a = 0$ and $b = \pi_p$ we obtain

$$\int_0^{\sin_p(x)} \frac{ds}{\sqrt[p]{\left(1-\frac{s^p}{p-1}\right)}} = x, \quad \text{for } x \in [0, \pi_p/2],$$

that is, $\sin_p = \zeta^{-1}$ where

$$\zeta(z) := \int_0^z \frac{ds}{\sqrt[p]{\left(1-\frac{s^p}{p-1}\right)}}, \quad \text{for } z \in \left[0, \sqrt[p]{p-1}\right].$$

With this definition, we extend $\sin_p$ to the interval $[\pi_p/2, \pi_p]$ symmetrically with respect to $\pi_p/2$ and afterward to the whole real line $\mathbb{R}$ as an odd, $2\pi_p$-periodic function. We list the basic properties of $\sin_p$:

1. $\sin_p(0) = 0 = \sin_p(\pi_p)$, $\sin_p(\pi_p/2) = \|\sin_p\|_\infty = \sqrt[p]{p-1}$.
2. $\sin_p(x)$ is strictly increasing in $[0, \pi_p/2]$ and strictly decreasing in $[\pi_p/2, \pi_p]$.
3. $\left|\sin_p'(x)\right| = \sqrt[p]{1 - \frac{|\sin_p|^p}{p-1}}$.

3 A sequence uniformly convergent to $\sin_p$

Let $I_p = [0, \pi_p/2]$ and define the following sequence of functions $\{\phi_n\} \subset C^1(I_p)$. Set $\phi_0 \equiv 1$ and

$$\begin{cases} \left(\psi_p\left(\phi_{n+1}'\right)\right)' = -\psi_p(\phi_n) & \text{if } x \in I_p, \\ \phi_{n+1}(0) = \phi_{n+1}'(\pi_p/2) = 0. \end{cases}$$

In this section, we prove that the scaled sequence $\left\{\sqrt[p]{p-1}\phi_n / \|\phi_n\|_\infty\right\}$ converges uniformly to $\sin_p$ in I_p. Before proceeding, we recall some basic properties of the ψ_p functions:

Proposition 3.1. (Basic properties of ψ_p) *The following holds:*

1. *ψ_p is continuous, strictly increasing and odd, for each $p > 1$.*
2. $\psi_p(ab) = \psi_p(a)\psi_p(b)$.
3. $\psi_p\left(\frac{a}{b}\right) = \frac{\psi_p(a)}{\psi_p(b)}$
4. $(\psi_p)^{-1} = \psi_{p'}$.
5. $\int_0^t \psi_p(s)\,ds = \frac{|t|^p}{p}$.

By a straightforward calculation we can find the following recursive integral expression for the ϕ_n-functions:

$$\phi_{n+1}(x) = \int_0^x \psi_{p'}\left(\int_\theta^{\pi_p/2} \psi_p(\phi_n(s))\,ds\right) d\theta. \tag{12}$$

It is clear from (12) that each ϕ_n is positive, increasing on I_p and reaches its maximum value at $x = \pi_p/2$. One can obtain an explicit expression for ϕ_1, the second function in the sequence:

$$\begin{aligned}\phi_1(x) &= \int_0^x \psi_{p'}\left(\int_\theta^{\pi_p/2} \psi_p(1)\, ds\right) d\theta \\ &= \int_0^x \psi_{p'}\left(\frac{\pi_p}{2} - \theta\right) d\theta \\ &= \int_{\pi_p/2-x}^{\pi_p/2} \psi_{p'}(y)\, dy \\ &= \frac{1}{p}\left[\left(\frac{\pi_p}{2}\right)^p - \left(\frac{\pi_p}{2} - x\right)^p\right].\end{aligned}$$

Note that

$$\|\phi_1\|_\infty = \phi_1\left(\frac{\pi_p}{2}\right) = \frac{1}{p}\left(\frac{\pi_p}{2}\right)^p = \frac{p-1}{p}\left(\frac{\pi/p}{\sin(\pi/p)}\right)^p.$$

The next ϕ_n-functions however, are very difficult to obtain explicitly by solving the integrals analytically. On the other hand, the integrals can easily be solved numerically.

Proposition 3.2. *$\phi_{n+1} \leqslant \|\phi_1\|_\infty \phi_n$ on I_p.*

Proof. For $n = 1$ the result is trivially true since $\phi_0 \equiv 1$.Assuming by induction that $\phi_n \leqslant \|\phi_1\|_\infty \phi_{n-1}$, we have

$$\begin{aligned}\phi_{n+1}(x) &= \int_0^x \psi_{p'}\left(\int_\theta^{\pi_p/2} \psi_p(\phi_n(s))\, ds\right) d\theta \\ &\leqslant \int_0^x \psi_{p'}\left(\int_\theta^{\pi_p/2} \psi_p(\|\phi_1\|_\infty \phi_{n-1}(s))\, ds\right) d\theta \\ &= \int_0^x \psi_{p'}\left(\psi_p(\|\phi_1\|_\infty) \int_\theta^{\pi_p/2} \psi_p(\phi_{n-1}(s))\, ds\right) d\theta \\ &= \|\phi_1\|_\infty \int_0^x \psi_{p'}\left(\int_\theta^{\pi_p/2} \psi_p(\phi_{n-1}(s))\, ds\right) d\theta \\ &= \|\phi_1\|_\infty \phi_n(x).\end{aligned}$$

■

The following technical lemma, which will be used in the sequel, can be proved via the Cauchy mean value theorem (see [AVV]) and works as a L'Hôpital's rule in order to get monotonicity for a certain quotient function.

Lemma 3.3. *Let $f, g : [a,b] \longrightarrow R$ be continuous on $[a,b]$ and differentiable in (a,b). Suppose $g'(x) \neq 0$ for all $x \in (a,b)$. If $\dfrac{f'}{g'}$ is (strictly) increasing [decreasing], then both $\dfrac{f(x)-f(a)}{g(x)-g(a)}$ and $\dfrac{f(x)-f(b)}{g(x)-g(b)}$ are (strictly) increasing [decreasing].*

Theorem 3.4. *For each $n \geqslant 1$ the function $\dfrac{\phi_n}{\phi_{n+1}}$ is strictly decreasing on I_p and*

(i) $\dfrac{1}{\|\phi_1\|_\infty} \leqslant \inf\limits_{I_p} \dfrac{\phi_n}{\phi_{n+1}} = \dfrac{\phi_n(\pi_p/2)}{\phi_{n+1}(\pi_p/2)} = \dfrac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}$.

(ii) $\left\|\dfrac{\phi_n}{\phi_{n+1}}\right\|_\infty = \psi_{p'}\left(\dfrac{\int_0^{\pi_p/2}\psi_p(\phi_{n-1}(s))\,ds}{\int_0^{\pi_p/2}\psi_p(\phi_n(s))\,ds}\right)$ for $n \geqslant 1$.

(iii) $\left\|\dfrac{\phi_n}{\phi_{n+1}}\right\|_\infty \leqslant \left\|\dfrac{\phi_{n-1}}{\phi_n}\right\|_\infty \leqslant \cdots \leqslant \left\|\dfrac{\phi_1}{\phi_2}\right\|_\infty < \infty$.

Proof. Since ϕ_1 is strictly increasing, it follows that $1/\phi_1$ is strictly decreasing. Assume by induction that ϕ_{n-1}/ϕ_n is strictly decreasing. Since

$$\frac{\phi_n(x)-\phi_n(0)}{\phi_{n+1}-\phi_{n+1}(0)} = \frac{\phi_n(x)}{\phi_{n+1}(x)},$$

in order to show that ϕ_n/ϕ_{n+1} is strictly decreasing, it suffices in light of the lemma to verify that ϕ_n'/ϕ_{n+1}' is strictly decreasing on I_p. But,

$$\frac{\phi_n'(x)}{\phi_{n+1}'(x)} = \frac{\psi_{p'}\left(\displaystyle\int_x^{\pi_p/2}\psi_p(\phi_{n-1}(s))\,ds\right)}{\psi_{p'}\left(\displaystyle\int_x^{\pi_p/2}\psi_p(\phi_n(s))\,ds\right)} = \psi_{p'}\left(\frac{\displaystyle\int_x^{\pi_p/2}\psi_p(\phi_{n-1}(s))\,ds}{\displaystyle\int_x^{\pi_p/2}\psi_p(\phi_n(s))\,ds}\right).$$

Since $\psi_{p'}$ is strictly increasing and the functions $\int_x^{\pi_p/2}\psi_p(\phi_{n-1}(s))\,ds$ and $\int_x^{\pi_p/2}\psi_p(\phi_n(s))\,ds$ are null at $x = \pi_p/2$, we can apply the lemma again to verify that the quotient of these integral functions is a strictly decreasing function. We have

$$\frac{\left(\displaystyle\int_x^{\pi_p/2}\psi_p(\phi_{n-1}(s))\,ds\right)'}{\left(\displaystyle\int_x^{\pi_p/2}\psi_p(\phi_n(s))\,ds\right)'} = \frac{\psi_p(\phi_{n-1}(s))}{\psi_p(\phi_n(s))} = \psi_p\left(\frac{\phi_{n-1}}{\phi_n}\right),$$

which is strictly decreasing by the induction hypothesis.

The inequality in (i) follows from Proposition 3.2. Before verifying (ii) we remark that $\|1/\phi_1\|_\infty = \infty$ since $\phi_1(0) = 0$. In order to prove (ii) we first observe that the monotonicity of ϕ_n/ϕ_{n+1} implies that

$$\left\|\frac{\phi_n}{\phi_{n+1}}\right\|_\infty = \lim_{x\to 0^+} \frac{\phi_n(x)}{\phi_{n+1}(x)}.$$

L'Hôpital's rule then yields

$$\lim_{x\to 0^+} \frac{\phi_n(x)}{\phi_{n+1}(x)} = \lim_{x\to 0^+} \frac{\phi_n'(x)}{\phi_{n+1}'(x)} = \psi_{p'}\left(\frac{\int_0^{\pi_p/2} \psi_p(\phi_{n-1}(s))\,ds}{\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,ds}\right) < \infty.$$

The proof of (iii) is a consequence of the following estimates, valid for $n \geqslant 2$:

$$\begin{aligned}
\left\|\frac{\phi_n}{\phi_{n+1}}\right\|_\infty &= \psi_{p'}\left(\frac{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_{n-1}(s))\,ds}{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,ds}\right) \\
&\leqslant \psi_{p'}\left(\frac{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,\psi_p\left(\frac{\phi_{n-1}}{\phi_n}(s)\right)ds}{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,ds}\right) \\
&\leqslant \psi_{p'}\left(\frac{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,\psi_p\left(\left\|\frac{\phi_{n-1}}{\phi_n}\right\|_\infty\right)ds}{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,ds}\right) \\
&= \left\|\frac{\phi_{n-1}}{\phi_n}\right\|_\infty \psi_{p'}\left(\frac{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,ds}{\displaystyle\int_0^{\pi_p/2} \psi_p(\phi_n(s))\,ds}\right) \\
&= \left\|\frac{\phi_{n-1}}{\phi_n}\right\|_\infty .
\end{aligned}$$

■

Theorem 2.4. *Let* $u_n := \dfrac{\phi_n}{\|\phi_n\|_\infty} \in C^1(I_p)$, *for* $n \geqslant 1$. *Then the sequence* $\{u_n(x)\}_{n\geqslant 1}$ *is decreasing for each* $x \in I_p$ *and*

$$\sqrt[p]{p-1}\,u_n \to \sin_p \quad \textit{uniformly in } I_p.$$

Proof. In I_p we have

$$\begin{aligned}\frac{u_n}{u_{n+1}} &= \frac{\phi_n}{\phi_{n+1}}\left(\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}\right)^{-1}\\ &\geqslant \left(\inf_{I_p}\frac{\phi_n}{\phi_{n+1}}\right)\left(\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}\right)^{-1}\\ &= \left(\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}\right)\left(\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}\right)^{-1}\\ &= 1,\end{aligned}$$

that is, $\{u_n(x)\}_{n\geqslant 1}$ is decreasing for each $x\in I_p$, and the whole sequence is bounded below by u_1. Thus, there exists

$$u := \lim u_n.$$

We have $\|u_n\|_\infty = 1$ for each n. Moreover, since

$$\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty} = \inf_{I_p}\frac{\phi_n}{\phi_{n+1}} \leqslant \left\|\frac{\phi_n}{\phi_{n+1}}\right\|_\infty \leqslant \left\|\frac{\phi_1}{\phi_2}\right\|_\infty =: C,$$

we also have, for every $x\in I_p$,

$$\begin{aligned}|u_n'(x)| &= \frac{1}{\|\phi_n\|_\infty}\psi_{p'}\left(\int_x^{\pi_p/2}\psi_p\left(\phi_{n-1}(s)\right)ds\right)\\ &= \frac{\|\phi_{n-1}\|_\infty}{\|\phi_n\|_\infty}\psi_{p'}\left(\int_x^{\pi_p/2}\psi_p\left(\frac{\phi_{n-1}(s)}{\|\phi_{n-1}\|_\infty}\right)ds\right)\\ &\leqslant C\psi_{p'}\left(\int_0^{\pi_p/2}\psi_p\left(u_{n-1}\right)ds\right)\\ &\leqslant C\psi_{p'}\left(\int_0^{\pi_p/2}\psi_p(1)\,ds\right)\\ &= \frac{C\pi_p}{2}.\end{aligned}$$

It follows from Arzela-Ascoli's theorem that $u_n \to u \in C(I_p)$, uniformly.

In order to conclude the proof, we need just to show that

$$u = \frac{\sin_p}{\sqrt[p]{p-1}}. \tag{13}$$

From (12) we can write the following expression:

$$u_{n+1}(x) = \gamma_n\int_0^x \psi_{p'}\left(\int_\theta^{\pi_p/2}\psi_p\left(u_n(s)\right)ds\right)d\theta,$$

where

$$\gamma_n := \frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}.$$

In view of the boundedness of $\{\gamma_n\}$, there exists $\gamma := \lim \gamma_{n_k}$ for some subsequence $\{\gamma_{n_k}\}$. Thus, letting $k \to \infty$ in

$$u_{n_k+1}(x) = \gamma_{n_k} \int_0^x \psi_{p'} \left(\int_\theta^{\pi_p/2} \psi_p \left(u_{n_k}(s) \right) ds \right) d\theta,$$

we get

$$u(x) = \gamma \int_0^x \psi_{p'} \left(\int_\theta^{\pi_p/2} \psi_p \left(u(s) \right) ds \right) d\theta \in C^1(I_p),$$

which means that u is a positive solution to the following problem

$$\begin{cases} \psi_p(u')' = -\gamma \psi_p(u) & \text{if } x \in I_p, \\ u(0) = u'(\pi_p/2) = 0. \end{cases}$$

In view of the positivity of u, we can integrate the equation above multiplied by u' and proceed as in the derivation of (9) to find $\gamma = 1$. From this we conclude that in fact $\lim \gamma_n = 1$ (the whole sequence converges to the eingenvalue 1) and that $u = \lim u_n$ satisfies the same boundary value problem that $\sin_p / \|\sin_p\|$ does. Since both u and $\sin_p$ are positive and $\|u\|_\infty = \|\sin_p / \|\sin_p\|\|_\infty = 1$, we must have

$$u = \frac{\sin_p}{\|\sin_p\|}$$

whence (13) follows. ■

4 Numerical Results

Next we examine the computational time of each method. Computations were performed on a WindowsXP/Pentium 4-2.8GHz platform, using the GCC compiler. Although the method of computing $\sin_p$ by solving an ODE suggested in [BR2] (which we implemented by means of a standard Runge-Kutta fourth power method) is by far the fastest, the computational times of the other two methods are competitive, the inverse power method being on average more than twice as fast as the power series method of [Lindqvist] for values of p greater than 2. Also, the average number of 8 iterations that the inverse power method uses to obtain the same (and sometimes better; see Table 2) accuracy of the differential equation method of [BR2] is quite remarkable, specially taking into account that the functions ϕ_n converge to 0 rather rapidly. We emphasize that the computational time of the inverse power method is not the main subject of this presentation. The method demands the computation of double integrals at each iteration for each grid point. We opted for a classical, computationally easy to implement and reasonably fast method to compute these integrals, namely, the Simpson composite method. However, a greater effort spent in lessening the computational time of the numerical integrations certainly would be reflected in a substantial decrease in the time spent computing $\sin_p$ overall. Nevertheless, by considering the accuracy and the comparison scale among the three methods (on the range of miliseconds) we may say that the results presented in this paper validate the inverse power method as an effective and reasonably fast method for numerically obtaining $\sin_p$.

Below we present the average time spent in computing $\sin_p$ on the whole interval I_p divided in 101 grid points by each method for six values of p (the average was taken out of five computer runs); the stop criterion in each method was an error tolerance of 10^{-8} between successive iterations and less than 500 terms in the power series.

p	1.1	1.5	2.0	2.5	3.0	3.5
Inverse power method	21.5	32.1	1.1	37.7	37.8	31.7
Differential equation method	1.9	1.8	1.1	1.5	1.5	1.5
Power series	92.9	2.2	2.0	79.6	79.3	73.3

Table 1: Average time (in miliseconds) for the computation of $\sin_p$ on I_p for each method.

Besides the trivial point 0, the only point where the value of $\sin_p$ is exactly known is $\pi_p/2$, with $\sin_p(\pi_p/2) = \sqrt[p]{p-1}$. In the next table we present the computed value for $\sin_p(\pi_p/2)$ obtained using each method:

p	1.1	1.5	2.0	2.5	3.0	3.5
$\sqrt[p]{p-1}$	0.123285	0.629961	1	1.17608	1.25992	1.29926
Inverse power method	0.123285	0.629961	1	1.17608	1.25992	1.29926
Differential equation method	0.123285	0.629966	1.00017	1.17647	1.26044	1.29983
Power series	5.3×10^{128}	0.629961	1	1.17608	1.25993	1.29928

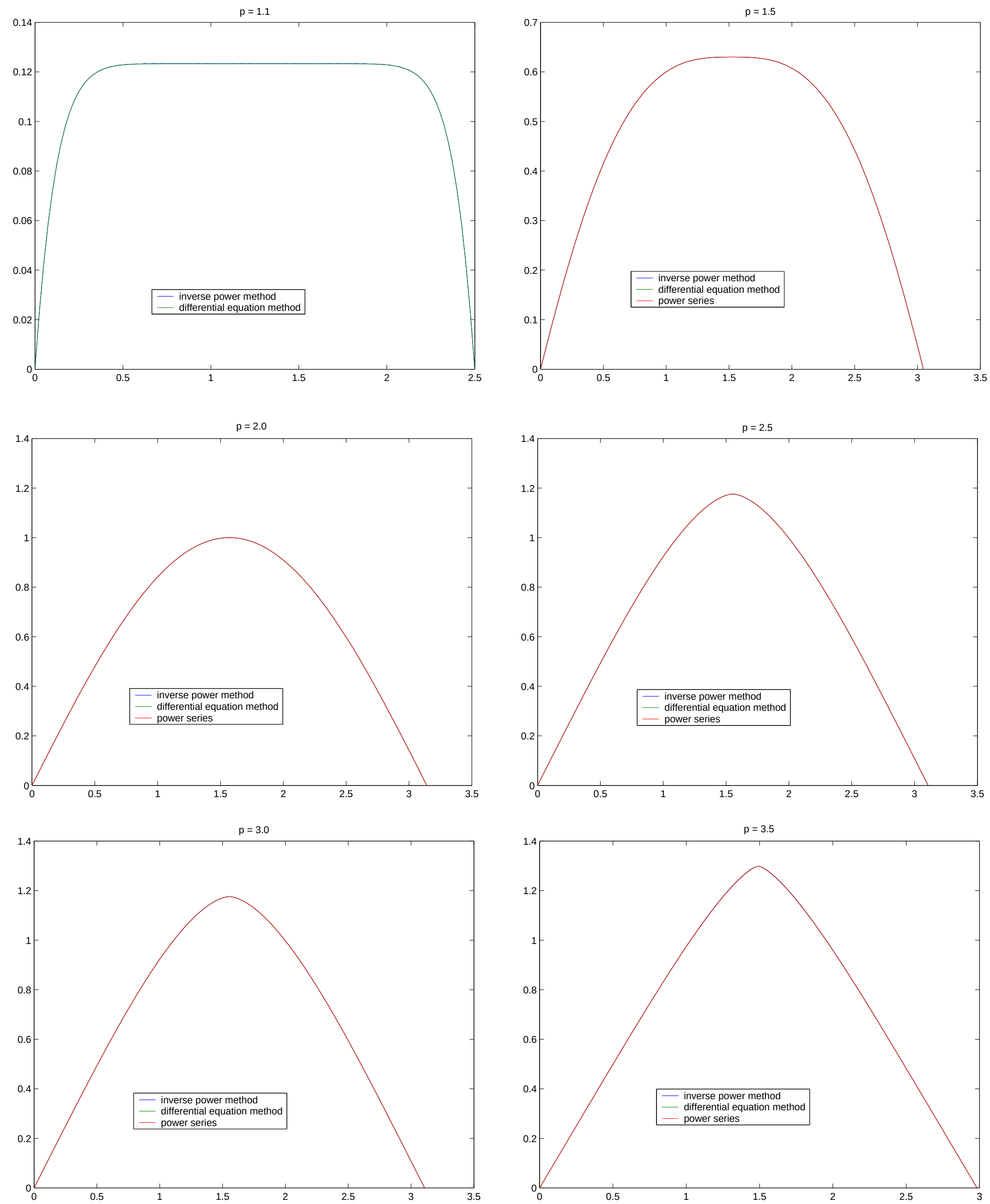
Table 2: Value of $\sin_p(\pi_p/2) = \sqrt[p]{p-1}$ obtained independently using each method.

Notice that the inverse power method appears to be more accurate when computing $\sin_p$ at values close to $\pi_p/2$. Indeed, in order to obtain a good approximation close to this point, it was necessary to allow for a greater number of terms in the power series than would be necessary for points far from $\pi_p/2$.

p	1.1	1.5	2.0	2.5	3.0	3.5
Inverse power method	5	8	9	8	8	8
Power Series	501	13	8	470	501	501

Table 3: Number of iterations.

We see that the number of iterations used by the inverse power method is remarkably low. Below, we present the graphics of $\sin_p$ for the same values of p computed using the three methods (except for $p = 1.1$, since the power series appears to diverge in this case). Notice that all three methods agree very well with each other, being virtually indistinguishable.



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