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Existence and multiplicity of positive solutions for the p-Laplacian with nonlocal coefficient ☆

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Abstract

We consider the Dirichlet problem with nonlocal coefficient given by $-a(\int_\Omega |u|^q\,dx)\Delta_p u = \mathrm{w}(x)f(u)$ in a bounded, smooth domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$), where Δ_p is the p-Laplacian, w is a weight function and the nonlinearity $f(u)$ satisfies certain local bounds. In contrast with the hypotheses usually made, no asymptotic behavior is assumed on f. We assume that the nonlocal coefficient $a(\int_\Omega |u|^q\,dx)$ ($q \geqslant 1$) is defined by a continuous and nondecreasing function $a : [0,\infty) \to [0,\infty)$ satisfying $a(t) > 0$ for $t > 0$ and $a(0) \geqslant 0$. A positive solution is obtained by applying the Schauder Fixed Point Theorem. The case $a(t) = t^{\gamma/q}$ ($0 < \gamma < p-1$) will be considered as an example where asymptotic conditions on the nonlinearity provide the existence of a sequence of positive solutions for the problem with arbitrarily large sup norm.

Keywords: p-Laplacian; Nonlocal coefficient; Existence and multiplicity of positive solutions

1. Introduction

In this paper we consider the quasilinear elliptic problem with nonlocal coefficient given by

$$\begin{cases} -a\left(\displaystyle\int_\Omega |u|^q\,dx\right)\Delta_p u = \mathrm{w}(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ ($p > 1$) is the p-Laplacian, $\Omega \subset \mathbb{R}^n$ ($n > 1$) is a bounded, smooth domain and both the weight function w and the nonlinearity f are nonnegative and continuous.

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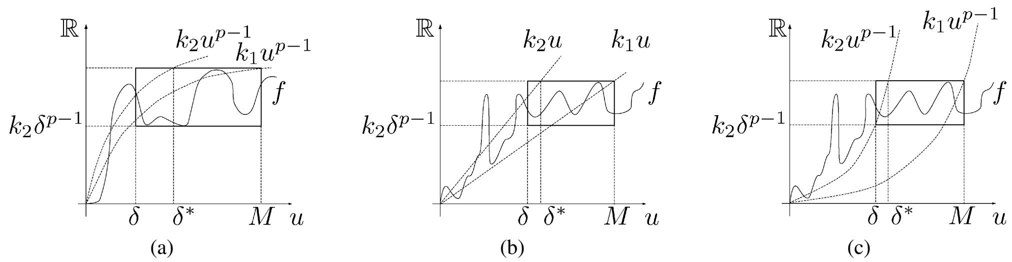

Fig. 1. The nonlinearity f passes through a "tunnel" Γ. The graph (a) displays the case $p < 2$, (b) the case $p = 2$ and (c) the case $p > 2$. Observe that the parameters $k_1 < k_2$ depend on δ and M, respectively.

We assume that the nonlocal coefficient $a(\int_\Omega |u|^q\,dx)$ $(q \geqslant 1)$ is defined by a continuous and nondecreasing function $a : [0,\infty) \to [0,\infty)$ satisfying

$$a(t) > 0 \quad \text{for all } t > 0 \quad \text{and} \quad a(0) \geqslant 0. \tag{2}$$

This kind of problem is considered, e.g., in [1,5]. In the last paper, the function a is supposed, additionally, to be strictly positive, that is, $a(0) > 0$. However, the arguments presented here allow us to treat also some instances where $a(0) = 0$ as, for example, $a(t) = t^{\gamma/q}$ $(0 < \gamma < p-1)$. This kind of nonlocal coefficient will be presented at the end of this paper as a prototype for producing examples not only of existence, but also of multiplicity of positive solutions for the nonlocal problem (1).

We assume, besides, that the weight $\mathrm{w} : \Omega \to [0,\infty)$ and the nonlinearity $f : [0,\infty) \to \mathbb{R}$ satisfy

(H1) Eventual zeroes of w are isolated: if $\mathrm{w}(x_0) = 0$, then there exists $\epsilon > 0$ such that

$$0 < |x - x_0| < \epsilon \quad \Rightarrow \quad \mathrm{w}(x) > 0. \tag{3}$$

(H2) There exist positive constants $\delta < M$ such that

$$\begin{cases} 0 \leqslant f(u) \leqslant k_1(\delta)M^{p-1} & \text{for } 0 \leqslant u \leqslant M, \\ k_2(M)\delta^{p-1} \leqslant f(u) & \text{for } \delta \leqslant u \leqslant M, \end{cases} \tag{4}$$

where $k_1(\delta) < k_2(M)$ are positive parameters, which also depend on the region Ω and the weight w, that will be defined later.

The last condition admits a geometric interpretation in terms of the graph of f in the u–v plane. It stays below of the horizontal line $v = k_1(M)$ for $u \in [0, M]$ and passes through a "tunnel" Γ defined by (see Fig. 1)

$$\Gamma = \big\{(u,v):\ \delta \leqslant u \leqslant M,\ k_2(M)\delta^{p-1} \leqslant v \leqslant k_1(\delta)M^{p-1}\big\}. \tag{5}$$

Our approach follows that introduced in [10,11] and further developed in [4] by using radial symmetrization techniques. However, the exposition here is self-contained.

2. Preliminaries

In this section B_R and B_α denote, respectively, balls centered at a fixed, but arbitrary point $x_0 \in \Omega$, whose radii are such that $B_\alpha \subset B_R \subset \Omega$. A suitable value of α will be defined later.

We intend to use the Schauder Fixed Point Theorem in the Banach space $X = C(\overline{\Omega})$ endowed with the sup norm. For this we define the subset Y of X, which depends on the positive values $\delta < M$ and on the balls $B_\alpha \subset B_R$:

$$Y = \left\{ u \in X:\ \begin{array}{ll} 0 \leqslant u(x) \leqslant M, & \text{if } x \in \overline{\Omega}, \\ \delta \leqslant u(x) \leqslant M, & \text{if } x \in B_\alpha(x_0) \subset \overline{\Omega}, \\ u(x) = 0, & \text{if } x \in \partial\Omega \end{array} \right\}. \tag{6}$$

It is clear that Y is a closed, convex and bounded subset of X. Moreover, if $u \in Y$, then

$$a\bigg(\int_\Omega |u|^q\,dx\bigg) \geqslant a\bigg(\int_{B_\alpha} |u|^q\,dx\bigg) \geqslant a\big(|B_\alpha|\delta^q\big) > 0 \tag{7}$$

and

$$a\left(\int_{\Omega}|u|^q\,dx\right)\leqslant a\left(M^q|\Omega|\right),\tag{8}$$

where $|\Omega|=\int_{\Omega}dx$.

Now, we define the operator $A:Y\to X$ that associates to each $u\in C(\overline{\Omega})$ the unique weak solution $v\in W_0^1(\overline{\Omega})\cap C^{1,\beta}(\overline{\Omega})\subset X$ of the Dirichlet problem

$$\begin{cases}-\Delta_p v(x)=\mathrm{w}(x)\dfrac{f(u(x))}{a(\int_{\Omega}|u|^q\,dx)}, & x\in\Omega,\\ v(x)=0, & x\in\partial\Omega.\end{cases}$$

We claim that the operator $A:X\to X$ is well defined, continuous and compact. In fact, it is well known that, to each $h\in L^{\infty}$, there exists a unique weak solution $v\in W_0^1(\overline{\Omega})\cap C^{1,\beta}(\overline{\Omega})$ of the Dirichlet problem $-\Delta_p v=h$ in Ω, for some $0<\beta<1$ (see [8], [13, Lemma 2] and [14] for interior estimates, and [12] for boundary estimates). As a consequence of the estimates of Lieberman and Tolksdorf [12,14], combined with the L^{∞}-estimates of Anane [2], we also have that $(-\Delta_p)^{-1}$ is continuous and compact on X. (This schematic proof presented here is proposed in [3].)

Coupling our claim with the properties of Y, specially (7), guarantees that A is a continuous and compact operator from Y to X.

To apply Schauder's Fixed Point Theorem we need to show that $A(Y)\subset Y$. In order to do that, we state some simple results.

We start by introducing a simple version of a useful comparison principle. General versions are established in [6, 7,9,13].

Lemma 1. *For $i\in\{1,2\}$, let $h_i\in C(\Omega)$ and $u_i\in C^{1,\alpha}(\overline{\Omega})$ be the weak solution of the problem $-\Delta_p u_i=h_i$ in Ω. If $h_1\leqslant h_2$ in Ω and $u_1\leqslant u_2$ in $\partial\Omega$, then $u_1\leqslant u_2$ in Ω.*

The second result concerns the solution of the radial Dirichlet problem

$$\begin{cases}\Delta_p u=h\left(|x-x_0|\right) & \text{in } B_R,\\ u=0 & \text{on } \partial B_R.\end{cases}\tag{9}$$

Lemma 2. *Suppose that $h\in C(\overline{B_R})$. Then, the (unique) solution of (9) is*

$$u(x)=\int_{|x-x_0|}^{R}\left(\int_0^{\theta}\left(\frac{s}{\theta}\right)^{n-1}h(s)\,ds\right)^{\frac{1}{p-1}}d\theta,\quad |x-x_0|\leqslant R.$$

Moreover, u belongs to $C^2(B_R)$, if $1<p\leqslant 2$, and, if $p>2$, u belongs to $C^{1,\beta}(B_R)$, where $\beta=1/(p-1)$.

Proof. It is straightforward to verify that the solution of (9) is the function stated. Regularity is trivial for $r=|x|>0$, and for $r=0$ we have

$$u'(0)=\lim_{r\to 0^+}\frac{u(r)-u(0)}{r}=-\lim_{r\to 0^+}\frac{1}{r}\int_0^r\left(\int_0^{\theta}\left(\frac{s}{\theta}\right)^{n-1}h(s)\,ds\right)^{\frac{1}{p-1}}d\theta=0$$

and

$$\lim_{r\to 0^+}\frac{u'(r)}{r^{\beta}}=-\lim_{r\to 0^+}\frac{1}{r^{\beta}}\left(\int_0^r\left(\frac{s}{r}\right)^{n-1}h(s)\,ds\right)^{\frac{1}{p-1}}=-\left(\frac{h(0)}{n}\lim_{r\to 0^+}r^{1-\beta(p-1)}\right)^{\frac{1}{p-1}},$$

for any $\beta>0$.

Therefore,

$$\lim_{r\to 0^+}\frac{u'(r)}{r^\beta}=\begin{cases}0 & \text{if } 1<p<2 \text{ and } \beta=1,\\ -\frac{h(0)}{n} & \text{if } p=2 \text{ and } \beta=1,\\ -\left(\frac{h(0)}{n}\right)^{\frac{1}{p-1}} & \text{if } p>2 \text{ and } \beta=\frac{1}{p-1}.\end{cases}$$

Lemma 1 implies the uniqueness of u. □

We are now in a position to define the parameters $k_1(\delta)$ and $k_2(M)$. For this, let $\phi\in C^{1,\beta}(\overline{\Omega})$ be the solution of

$$\begin{cases}-\Delta_p\phi=\mathrm{w} & \text{in } \Omega,\\ \phi=0 & \text{on } \partial\Omega.\end{cases}\tag{10}$$

Lemma 1 implies that $\phi\geqslant 0$ in Ω. However, taking into account the properties of w, we also have that $\|\phi\|_\infty>0$. Thus, we can define the positive parameter (that also depends on α)

$$k_1(\delta):=a\big(|B_\alpha|\delta^q\big)\|\phi\|_\infty^{1-p}\tag{11}$$

that appears in hypothesis (4). The value of α will be fixed later in (13), but we would like to emphasize that this value, as well as $\|\phi\|_\infty$, depends only on the region Ω and on the weight w.

Let Φ be the function defined by

$$\Phi:=\left(\frac{k_1(\delta)}{a(|B_\alpha|\delta^q)}\right)^{\frac{1}{p-1}}M\phi.$$

We observe that $0\leqslant\Phi(x)\leqslant M$ for all $x\in\Omega$, $\Phi\equiv 0$ on $\partial\Omega$ and

$$-\Delta_p\Phi(x)=\frac{k_1(\delta)M^{p-1}}{a(|B_\alpha|\delta^q)}\mathrm{w}(x),\quad \text{for all } x\in\Omega.$$

Moreover, it follows from (4) and (7) that, for any $u\in Y$,

$$-\Delta_p(Au)=\mathrm{w}(x)\frac{f(u)}{a(\int_\Omega|u|^q\,dx)}\leqslant\mathrm{w}(x)\frac{k_1(\delta)M^{p-1}}{a(|B_\alpha|\delta^q)}=-\Delta_p\Phi,\quad \text{for all } x\in\Omega,$$
$$(Au)(x)=0=\Phi(x),\quad \text{for all } x\in\partial\Omega.$$

Hence, Lemma 1 yields

$$0\leqslant Au\leqslant\Phi\leqslant M\quad \text{for all } u\in Y.\tag{12}$$

To define the parameter $k_2(M)$, we consider the radial symmetrization $\omega\in C[0,R]$ of the weight function w:

$$\omega(s)=\begin{cases}\min_{|y-x_0|=s}\mathrm{w}(y), & \text{if } 0<s\leqslant R,\\ \mathrm{w}(x_0), & \text{if } s=0.\end{cases}$$

We define $\alpha\in(0,R)$ by

$$\int_\alpha^R\left(\int_0^\alpha\left(\frac{s}{\theta}\right)^{n-1}\omega(s)\,ds\right)^{\frac{1}{p-1}}d\theta=\max_{0\leqslant r\leqslant R}\int_r^R\left(\int_0^r\left(\frac{s}{\theta}\right)^{n-1}\omega(s)\,ds\right)^{\frac{1}{p-1}}d\theta.\tag{13}$$

The right-hand side of the equality is nonnegative and vanishes at $r=0$ and $r=R$. Since $\omega(s)>0$ for all $s\in(0,\epsilon)$ and some $\epsilon>0$ sufficiently small (because the zeroes of w are isolated), this function attains a maximum value at the point that defines α. This value depends only on the region Ω and on the weight w, as claimed before.

Finally, we put

$$k_2(M):=a\big(|\Omega|M^q\big)\left[\int_\alpha^R\left(\int_0^\alpha\left(\frac{s}{\theta}\right)^{n-1}\omega(s)\,ds\right)^{\frac{1}{p-1}}d\theta\right]^{1-p}.\tag{14}$$

We observe that

$$\int_{\alpha}^{R}\left(\int_{0}^{\alpha}\left(\frac{s}{\theta}\right)^{n-1}\frac{\omega(s)k_2(M)}{a(|\Omega|M^q)}\,ds\right)^{\frac{1}{p-1}}d\theta=1 \tag{15}$$

and that

$$\frac{k_1(\delta)}{k_2(M)}=c^{p-1}\frac{a(|B_\alpha|\delta^q)}{a(|\Omega|M^q)}, \tag{16}$$

where

$$c:=\frac{\int_{\alpha}^{R}\big(\int_0^{\alpha}(\frac{s}{\theta})^{n-1}\omega(s)\,ds\big)^{\frac{1}{p-1}}\,d\theta}{\|\phi\|_\infty}. \tag{17}$$

We will now prove that

$$k_1(\delta)<k_2(M).$$

For this, we show that $a(|B_\alpha|\delta^q)\leqslant a(|\Omega|M^q)$ and that $c<1$.

The first inequality is an immediate consequence of the monotonicity of a, since $\delta<M$ and $B_\alpha\subset\Omega$ imply that

$$|B_\alpha|\delta^q=\delta^q\int_{B_\alpha}dx<M^q\int_{\Omega}dx=M^q|\Omega|.$$

To verify that $c<1$, let ϕ_R be defined by

$$\phi_R(x):=\int_{|x-x_0|}^{R}\left(\int_0^{\theta}\left(\frac{s}{\theta}\right)^{n-1}\omega(s)\,ds\right)^{\frac{1}{p-1}}d\theta.$$

We have that

$$-\Delta_p\phi_R=\omega\big(|x-x_0|\big)\leqslant \mathrm{w}(x)=-\Delta_p\phi,\quad \text{for all } x\in B_R,$$
$$\phi_R(x)=0\leqslant\phi(x),\quad \text{for all } x\in\partial B_R.$$

Hence, Lemma 1 yields $\phi_R\leqslant\phi$ in B_R and

$$\|\phi\|_\infty\geqslant\|\phi_R\|_\infty=\int_0^R\left(\int_0^{\theta}\left(\frac{s}{\theta}\right)^{n-1}\omega(s)\,ds\right)^{\frac{1}{p-1}}d\theta>\int_{\alpha}^{R}\left(\int_0^{\alpha}\left(\frac{s}{\theta}\right)^{n-1}\omega(s)\,ds\right)^{\frac{1}{p-1}}d\theta.$$

3. The theorem

In this section we prove the main result of this paper:

Theorem 3. *Suppose that f satisfies* (4) *for $k_1(\delta)$ and $k_2(M)$ defined by* (11) *and* (14)*, respectively. Then, the operator A has a fixed point $u\in C^{1,\beta}(\overline{\Omega})$, which is a solution of* (1) *satisfying*

$$\delta\leqslant\|u\|_\infty\leqslant M.$$

Proof. It suffices to show that the set Y defined by (6) is invariant under the operator A, since the result follows then from Schauder's Fixed Point Theorem.

Let $u\in Y$ and $v=Au\in C^{1,\beta}(\overline{\Omega})$. Since $v=0$ on $\partial\Omega$, the inequality (12) guarantees that we only have to show that $v\geqslant\delta$ in B_α. For this, as in [4], we define the auxiliary continuous function

$$h(r)=\begin{cases}\min_{|y-x_0|\leqslant r} f(u(y)), & \text{if } 0<r\leqslant R,\\ f(v(x_0)), & \text{if } r=0.\end{cases}$$

It is a consequence of the inequality (7) that h is well defined and that $h(|x-x_0|) \leqslant f(u(x))$ for all $x \in B_R$. Furthermore, for each $s \in (0,\alpha)$, there exists some $y_s \in B_s \subset B_\alpha$ such that

$$h(s) = f\big(u(y_s)\big).$$

Therefore, (4) and (7) imply that

$$h(s) \geqslant k_2(M)\delta^{p-1}, \quad \text{for all } s \in (0,\alpha). \tag{18}$$

Now, Lemma 2 guarantees that the nonnegative function defined by

$$z(x) := \int_{|x-x_0|}^{R} \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{n-1} \frac{\omega(s)h(s)}{a(M^q|\Omega|)}\, ds \right)^{\frac{1}{p-1}} d\theta, \quad \text{for } |x-x_0| \leqslant R,$$

satisfies

$$\begin{aligned} &-\Delta_p z = \frac{\omega(|x-x_0|)h(|x-x_0|)}{a(M^q|\Omega|)}, \quad \text{for } x \in B_R,\\ &z(x) = 0, \quad \text{for } x \in \partial B_R. \end{aligned} \tag{19}$$

We also have that $z \leqslant v$ in B_R. In fact, this follows from Lemma 1, since

$$-\Delta_p z = \frac{\omega h}{a(M^q|\Omega|)} \leqslant \frac{\mathrm{w} f(u)}{a(\int_\Omega |u|^q\, dx)} = -\Delta_p v \quad \text{in } B_R$$

and $z = 0 \leqslant v$ on $\partial\Omega$.

Because $z \leqslant v$ in B_R, we complete the proof by verifying that $\delta \leqslant z$ in $B_\alpha \subset B_R$. But this is a consequence of the definition of z and $k_2(M)$ and of the inequality (18) since, if $x \in B_\alpha$, we have that

$$\begin{aligned} z(x) &\geqslant \int_\alpha^R \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{n-1} \frac{\omega(s)h(s)}{a(M^q|\Omega|)}\, ds \right)^{\frac{1}{p-1}} d\theta \\ &\geqslant \int_\alpha^R \left(\int_0^\alpha \left(\frac{s}{\theta}\right)^{n-1} \frac{\omega(s)h(s)}{a(M^q|\Omega|)}\, ds \right)^{\frac{1}{p-1}} d\theta \\ &\geqslant \int_\alpha^R \left(\int_0^\alpha \left(\frac{s}{\theta}\right)^{n-1} \frac{\omega(s)k_2(M)\delta^{p-1}}{a(M^q|\Omega|)}\, ds \right)^{\frac{1}{p-1}} d\theta = \delta, \end{aligned}$$

the last equality being a consequence of (15). □

4. An example

In this section we present an explicit application of Theorem 3, by considering the nonlocal problem

$$\begin{cases} -\|u\|_q^\gamma \Delta_p u = \mathrm{w}(x)u^\beta & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

where $\|u\|_q$ denotes the L_q-norm in Ω. Here we have $a(t) = t^{\gamma/q}$ and $f(u) = u^\beta$. We claim that a necessary and sufficient condition for the application of Theorem 3 is

$$0 < \beta + \gamma < p - 1. \tag{21}$$

Since the nonlinearity u^β is increasing, in order to verify (4) it is sufficient to solve the system (in the unknowns δ and M)

$$\begin{cases} k_1(\delta)M^{p-1} = M^\beta,\\ k_2(M)\delta^{p-1} = \delta^\beta. \end{cases}$$

Taking into account the definitions of $k_1(\delta)$, $k_2(M)$ and the equality (16), we can write this system as

$$\begin{cases} |B_\alpha|^{\frac{\gamma}{q}}\delta^\gamma\|\phi\|_\infty^{1-p}M^{p-1} = M^\beta, \\ |\Omega|^{\frac{\gamma}{q}}M^\gamma\big(c\|\phi\|_\infty\big)^{1-p}\delta^{p-1} = \delta^\beta. \end{cases}$$

Dividing the first equation by the second, we find

$$\left(\frac{\delta}{M}\right)^{p-1-\gamma-\beta} = C(\mathrm{w}, \Omega),$$

where

$$C(\mathrm{w}, \Omega) := \left(\frac{|B_\alpha|}{|\Omega|}\right)^{\frac{\gamma}{q}} c^{p-1} < 1.$$

Therefore, since $\delta < M$, condition (21) is necessary and sufficient to obtain

$$\delta = \theta M$$

and

$$M = \big(\|\phi\|_\infty^{p-1}|B_\alpha|^{-\frac{\gamma}{q}}\theta^{-\gamma}\big)^{\frac{1}{p-1-\beta+\gamma}} = \big(|\Omega|^{-\frac{\gamma}{q}}\big(c\|\phi\|_\infty\big)^{p-1}\theta^{1-p+\beta}\big)^{\frac{1}{p-1-\beta+\gamma}},$$

where

$$\theta := C(\mathrm{w}, \Omega)^{\frac{1}{p-1-\beta-\gamma}}.$$

Therefore, it results from Theorem 3 the existence of at least one solution u of (20), with

$$\delta \leqslant \|u\|_\infty \leqslant M.$$

5. Multiplicity of solutions

It is clear that

$$k_2(M)\delta^{p-1} \leqslant k_1(\delta)M^{p-1} \quad \text{for } 0 < \delta < M, \tag{22}$$

is a necessary and sufficient condition for the existence of a "tunnel." We remark that, in the case of the equality $k_2(M) = k_1(\delta)$, the "tunnel" is degenerated, in the sense that it is a segment of the line. In this case, a nonlinearity f passes through it if, and only if, f is constant between δ and M.

As a consequence of (16), condition (22) is equivalent to

$$c^{p-1}\frac{a(|B_\alpha|\delta^q)}{a(|\Omega|M^q)} \geqslant \left(\frac{\delta}{M}\right)^{p-1} \quad \text{for } 0 < \delta < M, \tag{23}$$

where $c < 1$ is defined by (17). Therefore, it is evident that the existence of a "tunnel" is connected with properties of the nonlocal coefficient $a(t)$.

On the other hand, if $a \equiv 1$, the choice of $0 < \delta < M$ such that

$$0 < \frac{\delta}{M} \leqslant c^{p-1} < 1$$

always produces a "tunnel." By choosing sequences (δ_j) and (M_j) satisfying

$$\delta_j < M_j < \delta_{j+1} < M_{j+1} \quad \text{and} \quad \frac{\delta_j}{M_j} < N^{p-1}$$

we obtain a sequence of "tunnels" Γ_j such that $\Gamma_j \cap \Gamma_i = \emptyset$ if $j \neq i$. It is then easy to produce examples of nonlinearities $f(u)$ for which problem (1) has a sequence (u_j) of (distinct) solutions satisfying

$$\delta_j \leqslant \|u_j\|_\infty \leqslant M_j < \delta_{j+1} \leqslant \|u_{j+1}\|_\infty \leqslant M_{j+1}, \tag{24}$$

thus implying that $\|u_j\|_\infty \to \infty$.

For instance, such a nonlinearity can be chosen to be continuous, increasing and satisfying

$$\begin{cases} f(\delta_j) = k_2(M_j)\delta_j^{p-1}, \\ f(M_j) = k_1(\delta_j)M^{p-1}. \end{cases} \tag{25}$$

Since the graph of this function passes through all tunnels Γ_j we find, according to Theorem 3, a sequence (u_j) of positive solutions satisfying (24).

Let us now consider again the case $a(t) = t^{\gamma/q}$. Following the same reasoning just presented, denote $\mu = \delta/M \in (0, 1)$. Then, condition (23) can be written as

$$\mu^{p-1} \leqslant c^{p-1}\left(\frac{|B_\alpha|}{|\Omega|}\right)^{\frac{\gamma}{q}}\mu^\gamma.$$

Thus, if $0 < \gamma < p - 1$ and

$$\mu^{p-1-\gamma} \leqslant K_* := c^{p-1}\left(\frac{|B_\alpha|}{|\Omega|}\right)^{\frac{\gamma}{q}} \in (0, 1),$$

then there exists a sequence of disjunct tunnels Γ_j formed by sequences $\{\delta_j\}$ and $\{M_j\}$ satisfying

$$\delta_j < M_j < \delta_{j+1} < M_{j+1} \quad \text{and} \quad \frac{\delta_j}{M_j} < K_*.$$

As in the case $a \equiv 1$, these sequences can be used to produce a nonlinearity $f(u)$ such that the nonlocal problem

$$-\|u\|_q^\gamma \Delta_p u = \mathrm{w}(x)f(u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

has a sequence (u_j) of solutions satisfying (24). For this, it is enough to take a continuous and increasing function $f(u)$ satisfying (25).

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