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Factorization into k-bubbles for Palais–Smale maps to potential type energy functionals

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ABSTRACT

We prove a decomposition into generalized bubbles for Palais–Smale sequences associated with potential energy functionals for vector-valued function spaces. The study is motivated by the compactness question for solutions of critical potential systems, for which the existence problem was recently addressed. We also present some examples of the existence of radial generalized bubbles.

1. Introduction and main results

In 1984, Struwe established a compactness result for the well-known Brézis–Nirenberg problem [40]

$$\begin{cases} -\Delta u = |u|^{\frac{4}{n-2}} u + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a real parameter. Throughout this paper, $\Omega \subset \mathbb{R}^n$ denotes a smooth bounded domain for $n \geq 2$.

Our starting point is the following well-known existence result due to Brézis and Nirenberg [11].

Theorem A. *Let λ_1 be the first eigenvalue of the Laplace operator under the Dirichlet boundary condition. If $n \geq 4$ and $0 < \lambda < \lambda_1$, then (1) admits at least one positive solution.*

This is a central result in the theory of elliptic equations as it addresses the existence of solutions for boundary problems involving critical Sobolev growth, which in turn leads to a loss of compactness from a variational viewpoint.

Knowing Theorem A, Struwe investigated as a particular case the behavior of bounded solutions in $W_0^{1,2}(\Omega)$ for (1). Before we state his main result, we first describe some notations.

For each $1 < p < n$, the Sobolev space $W_0^{1,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}.$$

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The analogous form for the whole space, denoted by $\mathcal{D}^{1,p}(\mathbb{R}^n)$, is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx\right)^{1/p}.$$

Of course, we have $W_0^{1,p}(\Omega) \subset \mathcal{D}^{1,p}(\mathbb{R}^n)$.

Given sequences $(x_\alpha)_\alpha \in \overline{\Omega}$ and $(r_\alpha)_\alpha$ of positive numbers with the property $r_\alpha \to +\infty$ as $\alpha \to +\infty$, a 1-bubble is defined as a sequence $(B_\alpha)_\alpha$ of functions

$$B_\alpha(x) = (r_\alpha)^{\frac{n-2}{2}} u(r_\alpha(x - x_\alpha)),$$

obtained by renormalization of a nontrivial solution $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ of the equation

$$-\Delta u = |u|^{\frac{4}{n-2}} u \quad \text{in } \mathbb{R}^n. \tag{2}$$

We refer to x_α and r_α as the centers and weights of the 1-bubble $(B_\alpha)_\alpha$, respectively. We can write any positive solution u of (2) as [13,37]

$$u(x) = a^{\frac{n-2}{2}} u_0(a(x - x_0))$$

for all $a > 0$, where

$$u_0(x) = \left(1 + \frac{|x|^2}{n(n-2)}\right)^{-\frac{n-2}{2}}.$$

Struwe's main result [40] concerns decomposition into 1-bubbles for Palais–Smale sequences associated with the energy functional of (1), namely,

$$E_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx - \frac{n-2}{2n}\int_\Omega |u|^{\frac{2n}{n-2}}\, dx.$$

Thus, we have the following theorem.

Theorem B. *Let $n \geq 3$ and let $(u_\alpha)_\alpha$ be a non-negative Palais–Smale sequence to E_λ in $W_0^{1,2}(\Omega)$. Then there exists a solution $u^0 \in W_0^{1,2}(\Omega)$ of (1) and 1-bubbles $(B_\alpha^j)_\alpha$, $j = 1, \ldots, l$ such that some subsequence $(u_\alpha)_\alpha$ satisfies*

$$\left\| u_\alpha - u^0 - \sum_{j=1}^{l} B_\alpha^j \right\|_{D^{1,2}(\mathbb{R}^n)} \to 0 \quad \text{as } \alpha \to +\infty.$$

Subsequent to the work by Brézis and Nirenberg [11], much effort has been devoted to other questions and extensions of (1) [41, Chapter 3]. The literature contains many discussions of this issue [3,12,15–18,23,28,29,39].

A particular extension that has been extensively investigated is

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $1 < p < n$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator, and $p^* = \frac{np}{n-p}$ is the critical Sobolev exponent for embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$.

In 1987, Azorero and Peral extended Theorem A [4].

Theorem C. *Let $\lambda_{1,p}$ be the first eigenvalue of the p-Laplace operator under the Dirichlet boundary condition. If $n \geq p^2$ and $0 < \lambda < \lambda_{1,p}$, then (3) admits at least one positive solution.*

Several papers provide more details on the existence problem for (3) with $p \neq 2$ and other interesting questions [2,4,21,27,31].

Inspired by Theorem C, Mercuri and Willem [35] extended Theorem B to problems of the type (3). To state this, we consider again sequences $(x_\alpha)_\alpha \in \overline{\Omega}$ and $(r_\alpha)_\alpha$ of positive numbers such that $r_\alpha \to +\infty$ as $\alpha \to +\infty$. A 1-bubble of order p is simply a sequence $(B_\alpha)_\alpha$ of functions

$$B_\alpha(x) = (r_\alpha)^{\frac{n-p}{p}} u(r_\alpha(x - x_\alpha))$$

obtained by renormalization of a nontrivial solution $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ of the equation

$$-\Delta_p u = |u|^{p^*-2}u \quad \text{in } \mathbb{R}^n. \tag{4}$$

Analogous to the case $p = 2$, x_α and r_α denote the centers and weights, respectively, of the 1-bubble $(B_\alpha)_\alpha$ of order p. Solutions of (4) have been classified by Ghoussoub and Yuan [30] and by Damascelli and co-workers [19,20] for the special case of positive radial solutions. Precisely, any positive radial solution u of (4) is of the form

$$u(x) = \left(n \cdot a \left(\frac{n-p}{p-1} \right)^{p-1} \right)^{\frac{n-p}{p^2}} \left(a + |x|^{\frac{p}{p-1}} \right)^{-\frac{n-p}{p}}$$

for all constants $a > 0$.

The main result of Mercuri and Willem [35] concerns decomposition into 1-bubbles of order p for Palais–Smale sequences associated with the energy functional of (3):

$$E_{p,\lambda}(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p - \lambda |u|^p)\, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*}\, dx.$$

When the Palais–Smale sequence is non-negative, their main result yields the following theorem.

Theorem D. *Let $n \geq 2$, $1 < p < n$, and let $(u_\alpha)_\alpha$ be a non-negative Palais–Smale sequence to $E_{p,\lambda}$ in $W_0^{1,p}(\Omega)$. Then there exists a solution $u^0 \in W_0^{1,p}(\Omega)$ of (3) and 1-bubbles $(B_\alpha^j)_\alpha$ of order p, $j = 1, \ldots, l$, such that some subsequence $(u_\alpha)_\alpha$ satisfies*

$$\left\| u_\alpha - u^0 - \sum_{j=1}^{l} B_\alpha^j \right\|_{D^{1,p}(\mathbb{R}^n)} \to 0 \quad \text{as } \alpha \to +\infty.$$

Barbosa and Montenegro [5] established an extension of Theorem C dealing with potential (or gradient) elliptic systems, namely systems of the form

$$\begin{cases} -\Delta_p u = \dfrac{1}{p^*} \nabla F(u) + \dfrac{1}{p} \nabla G(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $1 < p < n$, $u = (u_1, \ldots, u_k)$, $\Delta_p u = (\Delta_p u_1, \ldots, \Delta_p u_k)$, and $F, G : \mathbb{R}^k \to \mathbb{R}$ are C^1 functions with F positive and homogeneous of degree p^* and G homogeneous of degree p. For physical reasons, the functions F and G are known in the literature as potential functions.

After a succession of papers addressed systems of the type (5) [1,7,22,36], Barbosa and Montenegro proved the following existence result that simultaneously extends Theorems A and C [5].

Theorem E. *Let $k \geq 1$ and let $F, G : \mathbb{R}^k \to \mathbb{R}$ be C^1 functions with F positive and homogeneous of degree p^* and G homogeneous of degree p. If $n \geq p^2$, $M_G := \max_{t \in \mathbb{S}_p^{k-1}} G(t) < \lambda_{1,p}$ and $G(t_0) > 0$ for some maximum point t_0 of F on $\mathbb{S}_p^{k-1} := \{t \in \mathbb{R}^k : |t|_p = 1\}$, then (5) admits at least one nontrivial solution.*

Barbosa and Montenegro also presented some classes of potential systems that admit non-negative solutions [5, Section 5]. By a non-negative map, we mean one in which each coordinate is non-negative.

When $k = 1$, note that (5) takes the form (3), since modulo constant factors $F(t) = |t|^{p^*}$ and $G(t) = \lambda |t|^p$. In particular, in this case, the conditions $M_G < \lambda_1$ and $G(t_0) > 0$ assumed in Theorem E correspond to $\lambda < \lambda_1$ and $\lambda > 0$, respectively.

When $k > 1$, there are many homogeneous potential functions. The following are canonical examples.

1. $F(t) = |t|_q^{p^*}$, $F(t) = |\pi_l(t)|^{\frac{p^*}{l}-1} \pi_l(t)$; and
2. $G(t) = |t|_q^p$, $G(t) = |\pi_l(t)|^{\frac{p}{l}-1} \pi_l(t)$, $G(t) = |\langle At, t \rangle|^{(p-2)/2} \langle At, t \rangle$,

where $|t|_q := (\sum_{i=1}^n |t_i|^q)^{1/q}$ is the Euclidean q-norm for $q \geq 1$, π_l is the lth elementary symmetric polynomial, $l = 1, \ldots, k$, $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product, and $A = (a_{ij})$ is a real $k \times k$ matrix.

Our main goal in this paper is to derive a compactness theorem for bounded non-negative solutions in the Sobolev k-space $W_0^{1,p}(\Omega, \mathbb{R}^k) := W_0^{1,p}(\Omega) \times \cdots \times W_0^{1,p}(\Omega)$ with respect to the product norm of (5) for the full range $1 < p < n$. For this, we introduce the notion of generalized bubbles, the so-called k-bubbles of order p, and prove a factorization into k-bubbles of order p for Palais–Smale sequences associated with the energy functional of (5). Our theorem works well for bounded non-negative solutions of a family of potential systems whose corresponding potential functions converge in some sense to F and G.

Consider the Sobolev k-space $\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k) := \mathcal{D}^{1,p}(\mathbb{R}^n) \times \cdots \times \mathcal{D}^{1,p}(\mathbb{R}^n)$ on $\mathbb{R}^n$ endowed with the product norm. Obviously, we have $W_0^{1,p}(\Omega, \mathbb{R}^k) \subset \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$. We begin by taking sequences $(x_\alpha)_\alpha \in \overline{\Omega}$ and $(r_\alpha)_\alpha$ of positive numbers satisfying $r_\alpha \to +\infty$ as $\alpha \to +\infty$. We define a k-bubble of order p as a sequence $(\mathcal{B}_\alpha)_\alpha$ of maps

$$\mathcal{B}_\alpha(x) = r_\alpha^{\frac{n-p}{p}} u(r_\alpha(x - x_\alpha)) \tag{6}$$

obtained by renormalization of a nontrivial solution $u = (u_1, \ldots, u_k) \in \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ of the system

$$-\Delta_p u = \frac{1}{p^*} \nabla F(u) \quad \text{in } \mathbb{R}^n. \tag{7}$$

As before, we call x_α and r_α the centers and weights, respectively, of the k-bubble $(\mathcal{B}_\alpha)_\alpha$ of order p.

Our main result establishes a decomposition into k-bubbles of order p for non-negative Palais–Smale sequences associated with the following energy functional of (5):

$$\mathcal{E}_{F,G}(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - G(u))\,dx - \frac{1}{p^*}\int_\Omega F(u)\,dx,$$

where

$$\int_\Omega |\nabla u|^p dx := \sum_{i=1}^k \int_\Omega |\nabla u_i|^p\,dx.$$

Theorem 1.1. *Let $k \geq 1, n \geq 2,\ 1 < p < n$, and $\mathbb{R}^k_+ := \{t \in \mathbb{R}^k : t_i \geq 0\}$. Let $F, G : \mathbb{R}^k \to \mathbb{R}$ be C^1 functions with F positive, even, homogeneous of degree p^* and, for some i, $D_iF(t) > 0$ for all $t \in \overline{\mathbb{R}}^k_+ \setminus \{0\}$, and G homogeneous of degree p. Let $(u_\alpha)_\alpha$ be a non-negative Palais–Smale sequence to $\mathcal{E}_{F,G}$ in $W^{1,p}_0(\Omega, \mathbb{R}^k)$. Then there exists a solution $u^0 \in W^{1,p}_0(\Omega, \mathbb{R}^k)$ of (5) and k-bubbles $(\mathcal{B}^j_\alpha)_\alpha$ of order $p, j = 1, \ldots, l$, such that some subsequence $(u_\alpha)_\alpha$ satisfies*

$$\left\| u_\alpha - u^0 - \sum_{j=1}^l \mathcal{B}^j_\alpha \right\|_{D^{1,p}(\mathbb{R}^n,\mathbb{R}^k)} \to 0 \quad \text{as } \alpha \to +\infty.$$

Theorem 1.1 is a complete extension of Theorems B and D. Following the ideas of Mercuri and Willem [35], it is possible to relax the assumption of non-negativity for $(u_\alpha)_\alpha$ by assuming only that the negative part of each component of $(u_\alpha)_\alpha$ converges to zero in $L^{p^*}(\Omega)$.

Note that Theorems B, D, and 1.1 provide compactness results for bounded sequences of non-negative solutions of (1), (3), and (5), respectively, since any such sequences are Palais–Smale sequences to each corresponding energy functional.

A more general fact for the compactness of the solutions can be stated as a consequence of Theorem 1.1.

Corollary 1.1. *Let $k \geq 1, n \geq 2$, and $1 < p < n$, and let $(F_\alpha)_\alpha$ and $(G_\alpha)_\alpha$ be sequences of C^1 functions on $\mathbb{R}^k$ converging to F and G in $C^1_{\text{loc}}(\mathbb{R}^k)$, respectively. Assume that F_α and F are homogeneous of degree p^*, F is even, positive and, for some i, satisfies $D_iF(t) > 0$ for all $t \in \mathbb{R}^k_+ \setminus \{0\}$, and G_α and G are homogeneous of degree p. Let $(u_\alpha)_\alpha \subset W^{1,p}_0(\Omega, \mathbb{R}^k)$ be a bounded sequence constructed from non-negative solutions u_α of the systems*

$$\begin{cases} -\Delta_p u = \dfrac{1}{p^*}\nabla F_\alpha(u) + \dfrac{1}{p}\nabla G_\alpha(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Then there exists a solution $u^0 \in W^{1,p}_0(\Omega, \mathbb{R}^k)$ of (5) and k-bubbles $(\mathcal{B}^j_\alpha)_\alpha$ of order $p, j = 1, \ldots, l$, such that some subsequence $(u_\alpha)_\alpha$ satisfies

$$\left\| u_\alpha - u^0 - \sum_{j=1}^l \mathcal{B}^j_\alpha \right\|_{D^{1,p}(\mathbb{R}^n,\mathbb{R}^k)} \to 0 \quad \text{as } \alpha \to +\infty.$$

The proof of this corollary is quite simple. It suffices to note that the convergence of $(F_\alpha)_\alpha$ and $(G_\alpha)_\alpha$ in $C^1_{\text{loc}}(\mathbb{R}^k)$ implies that $(u_\alpha)_\alpha$ is a Palais–Smale sequence to $\mathcal{E}_{F,G}$ in $W^{1,p}_0(\Omega, \mathbb{R}^k)$.

Compactness problems in PDEs still attract considerable interest, such as for singularly perturbed critical elliptic equations on bounded domains [14], critical anisotropic equations on bounded domains [32], critical elliptic equations on compact manifolds [25,38], critical potential systems on compact manifolds [24,26,33], and the Yamabe problem [8,9,34].

We conclude the paper with a classification result for certain solutions of (7), namely, those generated by solutions of (4). In other words, we provide an extension to $k > 1$ of the result established by Ghoussoub and Yuan for (4) [30]. Druet et al. determined an explicit form of the positive solutions (i.e., each positive coordinate) for $F(t) = \frac{1}{2^*}|t|_2^{2^*}$ [26]. For $k > 1$ and $p = 2$, Barbosa and Montenegro obtained a characterization of solutions of (7) that are extremal for a Sobolev inequality related to the potential F [6].

Theorem 1.2. *Let $k \geq 1, n \geq 2$, and $1 < p < n$, and let $F : \mathbb{R}^k \to \mathbb{R}$ be an even p^*-homogeneous positive C^1 function. Then (7) admits a nontrivial solution of the form tu, where $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$ and u is a nontrivial solution of (4), if and only if the vectors $t^p = (|t_1|^{p-2}t_1, \ldots, |t_k|^{p-2}t_k)$ and $\nabla F(t)$ are parallel. In this case, for any vector t_0 parallel to t there exists a radial solution u_0 of (7) satisfying $u_0(0) = t_0$. In particular, for $F(t) = \frac{1}{p^*}|t|_p^{p^*}$ and any vector $t_0 \in \mathbb{R}^k$, (7) admits a unique radial solution u_0 satisfying $u_0(0) = t_0$.*

Of course, there exist vectors $t \in \mathbb{R}^k$ such that t^p and $\nabla F(t)$ are parallel. To see this, it suffices to pick a maximum or minimum point t of the function F on the p-sphere $\mathbb{S}^{k-1}_p := \{t \in \mathbb{R}^k : |t|_p^p = \sum_{i=1}^k |t_i|^p = 1\}$, as can easily be seen from Lagrange multipliers.

The remainder of the paper is devoted to proofs of Theorems 1.1 and 1.2 in Sections 2 and 3, respectively.

2. Proof of Theorem 1.1

In this section we prove the decomposition into k-bubbles for Palais–Smale sequences associated with the energy functional $\mathcal{E}_{F,G}$ as described in the Introduction. We recall that a sequence $(u_\alpha)_\alpha$ in $W_0^{1,p}(\Omega, \mathbb{R}^k)$ is said to be Palais–Smale for $\mathcal{E}_{F,G}$ if

$$\mathcal{E}_{F,G}(u_\alpha) \text{ is bounded}$$

and

$$D\mathcal{E}_{F,G}(u_\alpha) \to 0 \quad \text{in } W_0^{1,p}(\Omega, \mathbb{R}^k)^*.$$

The proof of Theorem 1.1 requires the following seven steps.

Step 1. *Palais–Smale sequences for $\mathcal{E}_{F,G}$ are bounded in $W_0^{1,p}(\Omega, \mathbb{R}^k)$.*

Step 1 is used in the proof of the next step.

Step 2. *Let $(u_\alpha)_\alpha$ be a non-negative Palais–Smale sequence for $\mathcal{E}_{F,G}$. Then, up to a subsequence, $(u_\alpha)_\alpha$ converges weakly to u^0 in $W_0^{1,p}(\Omega, \mathbb{R}^k)$. Moreover, u^0 is a non-negative weak solution of (5).*

Step 3. *Let $\mathcal{I} : W_0^{1,p}(\Omega, \mathbb{R}^k) \to \mathbb{R}$ be the energy functional*

$$\mathcal{I}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \frac{1}{p^*}\int_\Omega F(u)\, dx$$

associated with the system

$$\begin{cases} -\Delta_p u = \dfrac{1}{p^*}\nabla F(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $(u_\alpha)_\alpha$ be a Palais–Smale sequence for $\mathcal{E}_{F,G}$ converging weakly to u^0 in $W_0^{1,p}(\Omega, \mathbb{R}^k)$. Then

$$\mathcal{E}_{F,G}(u_\alpha) = \mathcal{E}_{F,G}(u^0) + \mathcal{I}(u_\alpha - u^0) + o(1)$$

and $(u_\alpha - u^0)_\alpha$ is a Palais–Smale sequence for $\mathcal{I}$.

In what follows, we let $K_F(n, p)$ be a sharp constant for the potential-type Sobolev inequality

$$\left(\int_{\mathbb{R}^n} F(u)\, dx\right)^{\frac{1}{p^*}} \le K \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx\right)^{\frac{1}{p}}. \tag{9}$$

More precisely,

$$K_F(n, p) = \sup\left\{ \left(\int_{\mathbb{R}^n} F(u)\, dx\right)^{\frac{1}{p^*}} : u \in \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k),\ \|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^n,\mathbb{R}^k)} = 1 \right\}.$$

Barbosa and Montenegro proved that $K_F(n, p) = M_F^{\frac{1}{p^*}} K(n, p)$ [5], where M_F is the maximum of F on $\mathbb{S}_p^{k-1}$ and $K(n, p)$ is the sharp constant for the classical Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \le K \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

Step 4. *Let $(v_\alpha)_\alpha$ be a Palais–Smale sequence for $\mathcal{I}$ converging weakly to 0 in $W_0^{1,p}(\Omega, \mathbb{R}^k)$ such that $\mathcal{I}(v_\alpha) \to \beta$. If*

$$\beta < \beta^* := n^{-1} K_F(n, p)^{-n}$$

then $\beta = 0$ and $(v_\alpha)_\alpha$ converges strongly to 0 in $W_0^{1,p}(\Omega, \mathbb{R}^k)$.

Step 5. *Let $u^0 \in \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ be a nontrivial solution of the system (7). Then we have $\mathcal{J}(u^0) \ge \beta^*$, where $\mathcal{J} : \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k) \to \mathbb{R}$ denotes the energy functional given by*

$$\mathcal{J}(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p \, dx - \frac{1}{p^*}\int_{\mathbb{R}^n} F(u)\, dx.$$

Step 6. *Let $H=\{x\in\mathbb{R}^n : x_n>0\}$ and let $u\in\mathcal{D}_0^{1,p}(H,\mathbb{R}^k)$ be a non-negative weak solution of the potential system*

$$-\Delta_p u=\frac{1}{p^*}\nabla F(u)\quad in\ H,$$

where $\mathcal{D}_0^{1,p}(H,\mathbb{R}^k)$ denotes the completion of $C_0^\infty(H,\mathbb{R}^k)$ under the norm

$$\|u\|:=\left(\int_H|\nabla u|^p\,dx\right)^{1/p}.$$

Then $u\equiv 0$ on H.

Step 6 is used in the proof of the next step.

Step 7. *Let $(v_\alpha)_\alpha$ be a non-negative Palais–Smale sequence for $\mathcal{I}$ converging weakly to 0 in $W_0^{1,p}(\Omega,\mathbb{R}^k)$, but not strongly. Then there exists a sequence of points $(x_\alpha)_\alpha$ of Ω and a sequence of positive numbers $(r_\alpha)_\alpha$ with $r_\alpha\to+\infty$, a nontrivial solution v to (7) and a Palais–Smale sequence (w_α) for $\mathcal{I}$ in $W_0^{1,p}(\Omega,\mathbb{R}^k)$ such that, modulo a subsequence $(v_\alpha)_\alpha$, the following holds:*

$$w_\alpha(x)=v_\alpha(x)-\hat{B}_\alpha(x)+o(1),$$

where $\hat{B}_\alpha(x)=r_\alpha^{\frac{n-p}{p}}v(r_\alpha(x-x_\alpha))$ and $o(1)\to 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^n,\mathbb{R}^k)$. Moreover,

$$\mathcal{I}(w_\alpha)=\mathcal{I}(v_\alpha)-\mathcal{J}(v)+o(1)$$

and

$$r_\alpha \operatorname{dist}(x_\alpha,\partial\Omega)\to+\infty\quad \alpha\to+\infty.$$

For the moment, we postpone the proofs of Steps 1–7 to present the following proof.

Proof of Theorem 1.1. By Step 2, $(u_\alpha)_\alpha$ converges weakly to u^0 in $W_0^{1,p}(\Omega,\mathbb{R}^k)$; if $(u_\alpha)_\alpha$ converges strongly to u^0, the proof is complete. Otherwise, by [35, Lemma 3.5], without loss of generality we can consider that $(u_\alpha-u^0)_\alpha$ is non-negative, so we take the sequence $(v_\alpha^1)_\alpha$ given by $v_\alpha^1=u_\alpha-u^0$ and evoke Step 7 to find a sequence $(\mathcal{B}_\alpha^1)_\alpha$ of k-bubbles of order p such that the sequence $(v_\alpha^2)_\alpha$ defined by $v_\alpha^2=v_\alpha^1-\mathcal{B}_\alpha^1$ is Palais–Smale for $\mathcal{I}$. If $(v_\alpha^2)_\alpha$ converges strongly to 0 in $W_0^{1,p}(\Omega,\mathbb{R}^k)$, the proof is complete. Otherwise, we proceed inductively by letting

$$v_\alpha^1=u_\alpha-u^0\quad\text{and}\quad v_\alpha^j=u_\alpha-u^0-\sum_{i=1}^{j-1}\mathcal{B}_\alpha^i=v_\alpha^{j-1}-\mathcal{B}_\alpha^{j-1},$$

where $\mathcal{B}_\alpha^i=r_\alpha^{\frac{n-p}{p}}u^i(r_\alpha(\cdot-x_\alpha))$ and $u^i\in\mathcal{D}^{1,p}(\mathbb{R}^n,\mathbb{R}^k)$ is a nontrivial solution of (7). By Steps 3 and 5, we obtain

$$\mathcal{I}(v_\alpha^j)=\mathcal{E}_{F,G}(u_\alpha)-\mathcal{E}_{F,G}(u^0)-\sum_{i=1}^{j-1}\mathcal{J}(u^i)\le\mathcal{E}_{F,G}(u_\alpha)-\mathcal{E}_{F,G}(u^0)-(j-1)\beta^*.$$

We claim that this process stops after l steps. In fact, the preceding inequality and Step 4 furnish $\mathcal{I}(v_\alpha^{l+1})\le 0$ for some index $l\ge 0$. Thus, $v_\alpha^{l+1}=u_\alpha-u^0-\sum_{i=1}^{l}\mathcal{B}_\alpha^i$ converges strongly to 0 in $\mathcal{D}^{1,p}(\mathbb{R}^n,\mathbb{R}^k)$ and

$$\mathcal{E}_{F,G}(u_\alpha)-\mathcal{E}_{F,G}(u^0)-\sum_{i=1}^{l}\mathcal{J}(u^i)\to 0.\quad\square$$

Now we prove the seven steps.

Proof of Step 1. Let $(u_\alpha)_\alpha$ be a Palais–Smale sequence for $\mathcal{E}_{F,G}$. Thanks to the homogeneity properties satisfied by F and G, we derive

$$D\mathcal{E}_{F,G}(u_\alpha)\cdot u_\alpha=\int_\Omega\left(|\nabla u_\alpha|^p-G(u_\alpha)-F(u_\alpha)\right)\,dx=o(\|u_\alpha\|_{W^{1,p}(\Omega,\mathbb{R}^k)}),\tag{10}$$

so that

$$\mathcal{E}_{F,G}(u_\alpha)=\frac{1}{n}\int_\Omega F(u_\alpha)\,dx+\frac{1}{p}D\mathcal{E}_{F,G}(u_\alpha)\cdot u_\alpha=\frac{1}{n}\int_\Omega F(u_\alpha)\,dx+o(\|u_\alpha\|_{W^{1,p}(\Omega,\mathbb{R}^k)}).$$

Since $\mathcal{E}_{F,G}(u_\alpha)\le c$ for some constant $c>0$ independent of α, we obtain

$$\int_\Omega F(u_\alpha)\,dx\le nc+o\left(\|u_\alpha\|_{W^{1,p}(\Omega,\mathbb{R}^k)}\right).$$

Furthermore, since F is continuous, by Holder's inequality, we easily deduce that

$$\int_\Omega |u_\alpha|^p \, dx \le c + o\left(\|u_\alpha\|^{p/p^*}_{W^{1,p}(\Omega,\mathbb{R}^k)}\right),$$

where $c > 0$, like all the constants below, is independent of α. Writing

$$\int_\Omega \left(|\nabla u_\alpha|^p - G(u_\alpha)\right) \, dx = p\mathcal{E}_{F,G}(u_\alpha) + \frac{p}{p^*}\int_\Omega F(u_\alpha)\, dx,$$

we also obtain

$$\int_\Omega \left(|\nabla u_\alpha|^p - G(u_\alpha)\right) \, dx \le c + o\left(\|u_\alpha\|_{W^{1,p}(\Omega,\mathbb{R}^k)}\right).$$

Noting by the continuity of G that

$$\|u_\alpha\|^p_{W^{1,p}(\Omega,\mathbb{R}^k)} \le \int_\Omega \left(|\nabla u_\alpha|^p - G(u_\alpha)\right) \, dx + c\|u_\alpha\|^p_{L^p(\Omega,\mathbb{R}^k)},$$

it follows from the above equations that

$$\|u_\alpha\|^p_{W^{1,p}(\Omega,\mathbb{R}^k)} \le c + o\left(\|u_\alpha\|_{W^{1,p}(\Omega,\mathbb{R}^k)}\right) + o\left(\|u_\alpha\|^{p/p^*}_{W^{1,p}(\Omega,\mathbb{R}^k)}\right).$$

However, this clearly implies that $(u_\alpha)_\alpha$ is bounded in $W^{1,p}_0(\Omega,\mathbb{R}^k)$, which completes the proof of Step 1. □

Proof of Step 2. By Step 1 and the Sobolev embedding theorems, modulo a subsequence $u_\alpha \rightharpoonup u^0$ in $W^{1,p}_0(\Omega,\mathbb{R}^k)$ and $u_\alpha \to u^0$ in $L^q(\Omega,\mathbb{R}^k)$ for all $q < p^*$, where $L^q(\Omega,\mathbb{R}^k) := L^q(\Omega) \times \cdots \times L^q(\Omega)$, is endowed with the product norm.

Since $(u_\alpha)_\alpha$ is a Palais–Smale sequence, we have

$$\sum_{i=1}^k \int_\Omega |\nabla u^i_\alpha|^{p-2}\langle \nabla u^i_\alpha, \nabla\varphi_i\rangle \, dx - \int_\Omega \nabla G(u_\alpha)\cdot\varphi \, dx - \int_\Omega \nabla F(u_\alpha)\cdot\varphi\, dx = o(1) \tag{11}$$

for all $\varphi = (\varphi_1,\dots,\varphi_k) \in C^\infty_0(\Omega,\mathbb{R}^k)$, where $u_\alpha = (u^1_\alpha,\dots,u^k_\alpha)$. The strong convergence of $(u_\alpha)_\alpha$ in $L^q(\Omega,\mathbb{R}^k)$ and the regularity and homogeneity conditions on F and G yield

$$\int_\Omega \nabla F(u_\alpha)\cdot\varphi\, dx \to \int_\Omega \nabla F(u^0)\cdot\varphi\, dx$$

and

$$\int_\Omega \nabla G(u_\alpha)\cdot\varphi\, dx \to \int_\Omega \nabla G(u^0)\cdot\varphi\, dx$$

as $\alpha \to +\infty$. Conversely, the convergence of the first term of (11) is standard [38, Step 1.2 of Theorem 0.1]. Thus, we conclude from (11) that u^0 is a weak solution of (5) and it is straightforward to show that u^0 is non-negative. □

Proof of Step 3. A standard fact is that $|\nabla u^i_\alpha|^p \to |\nabla u^{0i}|^p$ a.e. in Ω for all i [38, Step 1.2 of Theorem 0.1], so by the Brézis–Lieb lemma [10] we have

$$\int_\Omega |\nabla u_\alpha|^p\, dx = \int_\Omega |\nabla(u_\alpha - u^0)|^p\, dx + \int_\Omega |\nabla u^0|^p\, dx + o(1). \tag{12}$$

According to the compactness,

$$\int_\Omega G(u_\alpha)\, dx = \int_\Omega G(u_\alpha - u^0)\, dx + \int_\Omega G(u^0)\, dx + o(1) \tag{13}$$

and by a version of the Brézis–Lieb lemma for maps [5],

$$\int_\Omega F(u_\alpha)\, dx = \int_\Omega F(u_\alpha - u^0)\, dx + \int_\Omega F(u^0)\, dx + o(1). \tag{14}$$

By setting $v_\alpha = u_\alpha - u^0$ and using (12)–(14), we can write

$$\begin{aligned}\mathcal{E}_{F,G}(u_\alpha) &= \frac{1}{p}\int_\Omega \left(|\nabla(v_\alpha + u^0)|^p - G(v_\alpha + u^0)\right)\, dx - \frac{1}{p^*}\int_\Omega F(v_\alpha + u^0)\, dx \\ &= \frac{1}{p}\int_\Omega \left(|\nabla v_\alpha|^p + |\nabla u^0|^p - G(v_\alpha) - G(u^0)\right)\, dx - \frac{1}{p^*}\int_\Omega \left(F(v_\alpha) + F(u^0)\right)\, dx + o(1) \\ &= \mathcal{E}_{F,G}(u^0) + \mathcal{I}(v_\alpha) + \frac{1}{p}\int_\Omega G(v_\alpha)\, dx + o(1).\end{aligned}$$

By the compactness and assumptions for G, the integral on the right-hand side goes to 0. In particular,

$$\mathcal{E}_{F,G}(u_\alpha) = \mathcal{E}_{F,G}(u^0) + \mathcal{I}(v_\alpha) + o(1). \tag{15}$$

To show that $(v_\alpha)_\alpha$ is a Palais–Smale sequence for $\mathcal{I}$, we note first that

$$\mathcal{I}(v_\alpha) = \mathcal{E}_{F,G}(u_\alpha) - \mathcal{E}_{F,G}(u^0) + o(1) = O(1) + o(1)$$

implies the boundedness of $(\mathcal{I}(v_\alpha))_\alpha$. Arguing as in Step 2, we have

$$\int_\Omega \nabla F(u_\alpha) \cdot \varphi \, dx = \int_\Omega \nabla F(u^0) \cdot \varphi \, dx + o(1) \tag{16}$$

and

$$\int_\Omega \nabla G(u_\alpha) \cdot \varphi \, dx = \int_\Omega \nabla G(u^0) \cdot \varphi \, dx + o(1) \tag{17}$$

for any $\varphi \in C_0^\infty(\Omega, \mathbb{R}^k)$.

Combining Eqs. (12)–(14), (16) and (17), we compute

$$\begin{aligned}
&D\mathcal{E}_{F,G}(v_\alpha + u^0) \cdot \varphi - D\mathcal{I}(v_\alpha) \cdot \varphi \\
&\quad = \sum_{i=1}^k \int_\Omega \left(|\nabla(v_\alpha^i + u^{0i})|^{p-2} \langle \nabla(v_\alpha^i + u^{0i}), \nabla\varphi_i \rangle \right) - \nabla G(v_\alpha^i + u^{0i}) \cdot \varphi_i \, dx - \int_\Omega \nabla F(v_\alpha + u^0) \cdot \varphi \, dx \\
&\qquad - \sum_{i=1}^k \int_\Omega |\nabla v_\alpha^i|^{p-2} \langle \nabla v_\alpha^i, \nabla \Phi_i \rangle \, dx + \int_\Omega \nabla F(v_\alpha) \cdot \varphi \, dx \\
&\quad = \sum_{i=1}^k \int_\Omega \langle |\nabla v_\alpha^i|^{p-2} \nabla u^{0i} + |\nabla u^{0i}|^{p-2} \nabla v_\alpha^i, \nabla\varphi_i \rangle \, dx \\
&\qquad + \sum_{i=1}^k \int_\Omega |\nabla u^{0i}|^{p-2} \langle \nabla u^{0i}, \nabla\varphi_i \rangle \, dx - \int_\Omega \nabla G(u^0) \cdot \varphi \, dx - \int_\Omega \nabla F(u^0) \cdot \varphi \, dx + o\left(\|\varphi\|_{W_0^{1,p}(\Omega, \mathbb{R}^k)} \right).
\end{aligned}$$

Using the fact that u^0 is a weak solution of (5), we can derive the desired result. □

Proof of Step 4. By Step 1, it follows that $(v_\alpha)_\alpha$ is bounded in $W_0^{1,p}(\Omega, \mathbb{R}^k)$. Then we can write

$$D\mathcal{I}(v_\alpha) \cdot v_\alpha = \int_\Omega |\nabla v_\alpha|^p \, dx - \int_\Omega F(v_\alpha) \, dx = o(1)$$

and

$$\mathcal{I}(v_\alpha) = \frac{1}{p} \int_\Omega |\nabla v_\alpha|^p \, dx - \frac{1}{p^*} \int_\Omega F(v_\alpha) \, dx = \beta + o(1).$$

From these relations, we obtain

$$\int_\Omega F(v_\alpha) \, dx = n\beta + o(1)$$

and

$$\int_\Omega |\nabla v_\alpha|^p \, dx = n\beta + o(1).$$

In particular, we derive $\beta \geq 0$. By its compactness, we can assume that $v_\alpha \to 0$ in $L^p(\Omega, \mathbb{R}^k)$. The F-Sobolev inequality [5]

$$\left(\int_\Omega F(v_\alpha) \, dx \right)^{\frac{p}{p^*}} \leq K_F(n, p)^p \int_\Omega |\nabla v_\alpha|^p \, dx$$

leads to

$$(n\beta)^{\frac{p}{p^*}} \leq K_F(n, p)^p n\beta.$$

We assert that $\beta = 0$. Assume, by contradiction, that $\beta > 0$. Then

$$(n\beta)^{\frac{p}{p^*}-1} = (n\beta)^{-\frac{p}{n}} \le K_F(n,p)^p,$$

so that

$$K_F(n,p)^p = (n\beta^*)^{-\frac{p}{n}} < (n\beta)^{-\frac{p}{n}} \le K_F(n,p)^p.$$

Since $\beta = 0$, we have

$$\int_\Omega |\nabla v_\alpha|^p \, dx = o(1).$$

In other words, $(v_\alpha)_\alpha$ converges to 0 in $W_0^{1,p}(\Omega, \mathbb{R}^k)$, which completes the proof of Step 4. □

Proof of Step 5. Let $u^0 \in \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ be a nontrivial solution of (7) Then, it follows directly from (7) that

$$\int_{\mathbb{R}^n} |\nabla u^0|^p \, dx = \int_{\mathbb{R}^n} F(u^0)\, dx \le K_F(n,p)^{p^*} \left(\int_{\mathbb{R}^n} |\nabla u^0|^p \, dx \right)^{\frac{p^*}{p}}.$$

However, this clearly implies

$$\mathcal{J}(u^0) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u^0|^p \, dx - \frac{1}{p^*} \int_{\mathbb{R}^n} F(u^0)\, dx \ge \frac{1}{n} K_F(n,p)^{-n} = \beta^*. \quad \square$$

We prove Step 6 using two lemmas. The first is the following weakened form of the divergence theorem presented by Mercuri and Willem [35].

Lemma 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with outer normal unit vector $\nu(\cdot)$ and let $v \in C(\mathbb{R}^n, \mathbb{R}^n)$ be such that* $\operatorname{div} v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. *Then*

$$\int_\Omega \operatorname{div} v \, dx = \int_{\partial\Omega} v(\sigma) \cdot \nu(\sigma)\, d\sigma.$$

Hereafter we denote $H = \{x \in \mathbb{R}^n : x_n > 0\}$. The next lemma is inspired by Mercuri and Willem [35] and its proof proceeds in the same spirit.

Lemma 2.2. *Let $k \ge 1$, $n \ge 2$, and $1 < p < n$, and let $F : \mathbb{R}^k \to \mathbb{R}$ be a function of the C^1 class that is positive, even, and homogeneous of degree p^*. Let $u \in \mathcal{D}_0^{1,p}(H, \mathbb{R}^k)$ be weak solution of the system*

$$-\Delta_p u = \frac{1}{p^*} \nabla F(u) \quad \text{in } H. \tag{18}$$

Then $D_n u := \frac{\partial}{\partial x_n} u = 0$ everywhere on ∂H.

Proof. Let $u \in \mathcal{D}_0^{1,p}(H, \mathbb{R}^k)$ be a weak solution of (18). By the anti-reflection of u in $\mathbb{R}^n \setminus H$ with respect to ∂H, we can extend u to a map $v \in \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$. Since F is even, it follows that v is a weak solution of (7). By [5, Lemma 2.5], we have $v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^k)$. Thus, since $|\nabla F(v)| \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, by [42, Proposition 1] we find $v \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^k)$ with $q = \min\{p, 2\}$. In particular,

$$-\Delta_p v = \frac{1}{p^*} \nabla F(v)$$

almost everywhere in $\mathbb{R}^n$. Thus, we can easily derive

$$\operatorname{div}(D_n v_i |\nabla v_i|^{p-2} \nabla v_i) = D_n v_i \, \Delta_p v_i + |\nabla v_i|^{p-2} \nabla v_i \cdot \nabla(D_n v_i) \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Let B_ρ be a ball of center 0 and radius ρ in $\mathbb{R}^n$. From Lemma 2.1, we have

$$\begin{aligned} \int_{H\cap B_\rho} D_n u_i \operatorname{div}(|\nabla u_i|^{p-2} \nabla u_i)\, dx &= \int_{\partial(H\cap B_\rho)} D_n u_i \, |\nabla u_i|^{p-2} \nabla u_i \cdot \nu(\sigma)\, d\sigma - \int_{H\cap B_\rho} |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla(D_n u_i)\, dx \\ &= \int_{\partial(H\cap B_\rho)} D_n u_i \, |\nabla u_i|^{p-2} \nabla u_i \cdot \nu(\sigma)\, d\sigma - \int_{\partial(H\cap B_\rho)} \frac{|\nabla u_i|^p}{p} \nu_n(\sigma)\, d\sigma \end{aligned}$$

and

$$\frac{1}{p^*} \int_{H\cap B_\rho} \nabla F(u) \cdot D_n u \, dx = \frac{1}{p^*} \int_{\partial(H\cap B_\rho)} F(u) \nu_n(\sigma)\, d\sigma = \frac{1}{p^*} \int_{H\cap \partial B_\rho} F(u) \nu_n(\sigma)\, d\sigma.$$

Let $X = (|\nabla u_1|^{p-2}\nabla u_1 \cdot \nu, \dots, |\nabla u_k|^{p-2}\nabla u_k \cdot \nu)$. Thanks to (18), we obtain

$$\left(1-\frac{1}{p}\right)\int_{\partial H\cap B_\rho} |D_n u|^p \, d\sigma = \int_{H\cap\partial B_\rho} D_n u \cdot X(\sigma)\, d\sigma - \int_{H\cap\partial B_\rho} \frac{|\nabla u|^p}{p}\nu_n(\sigma)\, d\sigma + \int_{H\cap\partial B_\rho} F(u)\nu_n(\sigma)\, d\sigma.$$

Note that the right-hand side is bounded by

$$M(\rho) = \left(1+\frac{1}{p}\right)\int_{H\cap\partial B_\rho} \frac{|\nabla u|^p}{p}\, d\sigma + \int_{H\cap\partial B_\rho} F(u)\, d\sigma.$$

Since $\nabla u \in L^p(H, \mathbb{R}^k)$ and $u \in L^{p^*}(H, \mathbb{R}^k)$, there exists a sequence $\rho_\alpha \to \infty$ such that $M(\rho_\alpha) \to 0$. The monotone convergence theorem then furnishes $\int_{\partial H} |D_n u|^p \, d\sigma = 0$, which concludes the proof of Lemma 2.2. □

Proof of Step 6. Assume that u is a nontrivial non-negative weak solution. Since $D_i F(u) > 0$ and $u_i \geq 0$, we obtain $\Delta_p u_i \leq 0$ and $\Delta_p u_i \not\equiv 0$. Since $u \in C^{1,\alpha}_{\mathrm{loc}}(\overline{H}, \mathbb{R}^k)$, by the strong maximum principle [43] we obtain $D_n u_i > 0$ on ∂H. Conversely, by Lemma 2.2 we have $D_n u_i = 0$ on ∂H. This contradiction leads us to the conclusion of Step 6. □

Proof of Step 7. We prove Step 7 using three lemmas that are introduced during the proof. First, up to a subsequence, we can assume that $\mathcal{I}(v_\alpha) \to \beta$ as $\alpha \to +\infty$. Moreover, by the density of $C_0^\infty(\Omega, \mathbb{R}^k)$ in $W_0^{1,p}(\Omega, \mathbb{R}^k)$, we assume that each map v_α is smooth. Since $D\mathcal{I}(v_\alpha) \to 0$,

$$\frac{1}{n}\int_\Omega |\nabla v_\alpha|^p \, dx = \mathcal{I}(v_\alpha) - \frac{1}{p^*} D\mathcal{I}(v_\alpha) \cdot v_\alpha \to \beta,$$

and hence, by Step 4,

$$\liminf_{\alpha\to+\infty} \int_\Omega |\nabla v_\alpha|^p \, dx = n\beta \geq K_F(n,p)^{-n}. \tag{19}$$

For $t > 0$, let

$$\mu_\alpha(t) = \sup_{x\in\Omega}\left(\int_{B_t(x)} |\nabla v_\alpha|^p \, dx\right),$$

where $B_t(x)$ denotes the ball with radius t and center x in $\mathbb{R}^n$. It follows from (19) that $\mu_\alpha(t) > 0$ and $\lim_{t\to+\infty}\mu_\alpha(t) \geq K_F(n,p)^{-n}$. Let $0 < \delta < K_F(n,p)^{-n}$. Since v_α is smooth, $\mu_\alpha(\cdot)$ is continuous. Thus, for any $\lambda \in (0, \delta)$, there exists $t_\alpha \in (0, +\infty)$ such that $\mu_\alpha(t_\alpha) = \lambda$. There also exists $y_\alpha \in \overline{\Omega}$ such that

$$\int_{B_{t_\alpha}(y_\alpha)} |\nabla v_\alpha|^p \, dx = \lambda.$$

In conclusion, we can choose $x_\alpha \in \overline{\Omega}$ and r_α such that the rescaling

$$\tilde{v}_\alpha(x) = r_\alpha^{-\frac{n-p}{p}} v_\alpha\left(\frac{x}{r_\alpha} + x_\alpha\right)$$

satisfies

$$\tilde{\mu}_\alpha(1) = \sup_{\substack{x\in\mathbb{R}^n \\ \frac{x}{r_\alpha}+x_\alpha\in\Omega}} \int_{B_1(x)} |\nabla \tilde{v}_\alpha|^p \, dx = \int_{B_1(0)} |\nabla \tilde{v}_\alpha|^p \, dx = \frac{1}{2L} K_F(n,p)^{-n}, \tag{20}$$

where $L \in \mathbb{N}$ is such that $B_2(0)$ is covered by L balls of radius 1 centered on $B_2(0)$. According to (19), there exists $r_0 > 0$ such that $r_\alpha \geq r_0$ for all α. Of course,

$$\|\tilde{v}_\alpha\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n,\mathbb{R}^k)} = \|v_\alpha\|^p_{W^{1,p}(\Omega,\mathbb{R}^k)} \to n\beta < \infty,$$

so that $\tilde{v}_\alpha \rightharpoonup \tilde{v}_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ up to a subsequence. Furthermore, by construction, $\tilde{v}_0 \geq 0$.

Our first lemma is as follows.

Lemma 2.3. *We have $\tilde{v}_\alpha \to \tilde{v}_0$ in $W^{1,p}(\Omega', \mathbb{R}^k)$ for any $\Omega' \subset\subset \mathbb{R}^n$.*

Proof. To prove this claim, it suffices to verify its validity for $\Omega' = B_1(x_0)$ for any $x_0 \in \mathbb{R}^n$. By Fubini's theorem, we have

$$\int_1^2 \left(\int_{\partial B_{\rho_\alpha}(x_0)} |\nabla \tilde{v}_\alpha|^p \, d\sigma\right) dr \leq \int_{B_2(x_0)} |\nabla \tilde{v}_\alpha|^p \, dx \leq n\beta + o(1).$$

By the mean value theorem, we obtain that there exists a radius $\rho_\alpha \in [1, 2]$ such that

$$\int_{\partial B_{\rho_\alpha}(x_0)} |\nabla \tilde{v}_\alpha|^p \, d\sigma \le 2n\beta + o(1). \tag{21}$$

Let $\hat{p} = \frac{p-1}{p}$ and $W^{\hat{p},p}(\partial\Omega, \mathbb{R}^k)$ be the space product $W^{\hat{p},p}(\partial\Omega, \mathbb{R}^k) = W^{\hat{p},p}(\partial\Omega) \times \cdots \times W^{\hat{p},p}(\partial\Omega)$ endowed with the product topology, where $W^{\hat{p},p}(\partial\Omega)$ denotes the space of the trace function in $W^{1,p}(\Omega)$. By the compactness of the embedding $W^{1,p}(\partial B_{\rho_\alpha}(x_0), \mathbb{R}^k) \hookrightarrow W^{\hat{p},p}(\partial B_{\rho_\alpha}(x_0), \mathbb{R}^k)$ [41, Appendix A], up to a subsequence we deduce that $\tilde{v}_\alpha$ converges strongly to $\overline{v}_0$ in $W^{\hat{p},p}(\partial B_{\rho_\alpha}(x_0), \mathbb{R}^k)$. In addition, by the compactness of the trace operator $W^{1,p}(B_{\rho_\alpha}(x_0), \mathbb{R}^k) \hookrightarrow L^p(\partial B_{\rho_\alpha}(x_0), \mathbb{R}^k)$, we have $\tilde{v}_0 = \overline{v}_0$. We define

$$\phi_\alpha = \begin{cases} \tilde{v}_\alpha - \tilde{v}_0 & \text{in } B_{\rho_\alpha}(x_0) \\ \tilde{w}_\alpha & \text{in } B_3(x_0) \setminus B_{\rho_\alpha}(x_0) \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{w}_\alpha$ denotes the solution of the Dirichlet problem

$$\begin{cases} \Delta_p \tilde{w}_\alpha = 0 & \text{in } B_3(x_0) \setminus B_{\rho_\alpha}(x_0) \\ \tilde{w}_\alpha = \tilde{v}_\alpha - \tilde{v}_0 & \text{on } \partial B_{\rho_\alpha}(x_0) \\ \tilde{w}_\alpha = 0 & \text{on } \partial B_3(x_0) \end{cases}.$$

The existence of such $\tilde{w}_\alpha$ is guaranteed [38, Step 2.2 of Lemma 1.1]. The same step guarantees the existence of a constant $c > 0$, independent of ρ_α, $\tilde{w_\alpha}$ and $\tilde{v_\alpha} - \tilde{v}_0$, such that

$$\|\tilde{w}_\alpha\|_{W^{1,p}(B_3(x_0)\setminus B_{\rho_\alpha}(x_0), \mathbb{R}^k)} \le C \, \|\tilde{v}_\alpha - \tilde{v}_0\|_{W^{\hat{p},p}(\partial B_{\rho_\alpha}(x_0), \mathbb{R}^k)},$$

which gives us

$$\|\tilde{w}_\alpha\|_{W^{1,p}(B_3(x_0)\setminus B_{\rho_\alpha}(x_0), \mathbb{R}^k)} \to 0. \tag{22}$$

Consider the rescaling $\hat{\phi}_\alpha(x) = r_\alpha^{\frac{n-p}{p}} \phi_\alpha(r_\alpha(x - x_\alpha))$. Since supp $\phi_\alpha \subset B_3(x_0)$ for α large enough, we obtain supp $\hat{\phi}_\alpha \subset B_{3r_\alpha^{-1}}\left(\frac{x_0}{r_\alpha} + x_\alpha\right) \subset \Omega$. Since $(v_\alpha)_\alpha$ is a Palais–Smale sequence for $\mathcal{I}$, we have

$$D\mathcal{J}(\tilde{v}_\alpha) \cdot \phi_\alpha = D\mathcal{I}(v_\alpha) \cdot \hat{\phi}_\alpha = o(1).$$

Thanks to the definition of ϕ_α, Eqs. (12) and (14), the assumptions on F, the strong convergence $\tilde{v}_\alpha \to \tilde{v}_0$ in $L^q(\Omega, \mathbb{R}^k)$ with $q < p^*$, and Eqs. (9) and (22), we deduce that

$$\begin{aligned} o(1) &= D\mathcal{J}(\tilde{v}_\alpha) \cdot \phi_\alpha \\ &= \sum_{i=1}^k \int_{\mathbb{R}^n} \left(|\nabla \tilde{v}_\alpha^i|^{p-2} \langle \nabla \tilde{v}_\alpha^i, \nabla \phi_\alpha^i \rangle - \frac{1}{p^*} \partial_i F(\tilde{v}_\alpha) \cdot \phi_\alpha^i \right) dx \\ &= \int_{B_{\rho_\alpha}(x_0)} \left(|\nabla(\tilde{v}_\alpha - \tilde{v}_0)|^p - F(\tilde{v}_\alpha - \tilde{v}_0) \right) dx + o(1) \\ &= \int_{\mathbb{R}^n} \left(|\nabla \phi_\alpha|^p - F(\phi_\alpha) \right) dx + o(1) \\ &\ge \|\phi_\alpha\|^p_{\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)} \left(1 - K_F(n,p)^{p^*} \|\phi_\alpha\|^{p^*-p}_{\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)} \right) + o(1), \end{aligned} \tag{23}$$

where $o(1) \to 0$ as $\alpha \to +\infty$. Conversely, by the definition of ϕ_α and Eqs. (12), (21), and (22),

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla \phi_\alpha|^p \, dx &= \int_{B_{\rho_\alpha}(x_0)} |\nabla(\tilde{v}_\alpha - \tilde{v}_0)|^p \, dx + \int_{B_3(x_0)\setminus B_{\rho_\alpha}(x_0)} |\nabla \tilde{w}_\alpha|^p \, dx + o(1) \\ &= \int_{B_{\rho_\alpha}(x_0)} |\nabla(\tilde{v}_\alpha - \tilde{v}_0)|^p \, dx + o(1) \\ &\le \int_{B_{\rho_\alpha}(x_0)} (|\nabla \tilde{v}_\alpha|^p - |\nabla \tilde{v}_0|^p) \, dx + o(1) \\ &\le \int_{B_2(x_0)} |\nabla \tilde{v}_\alpha|^p \, dx + o(1) \\ &\le L\tilde{\mu}_\alpha(1) = \frac{K_F(n,p)^{-n}}{2} + o(1). \end{aligned}$$

Therefore, from (23), we conclude that $\phi_\alpha \to 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$. In particular, $\tilde{v}_\alpha \to \tilde{v}_0$ in $W^{1,p}(B_1(x_0))$. $\square$

Given $\psi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^k)$, we then derive

$$D\mathcal{J}(\tilde{v}_0) \cdot \psi = \lim_{\alpha \to +\infty} D\mathcal{J}(\tilde{v}_\alpha) \cdot \psi = 0$$

so that $\tilde{v}_0 \in \mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ is a non-negative weak solution of (7). By Lemma 2.3 and (20), we find

$$\int_{B_1(0)} |\nabla \tilde{v}_0|^p \, dx = \frac{K_F(n,p)^{-n}}{2L} > 0,$$

so that $\tilde{v}_0 \not\equiv 0$ on $\mathbb{R}^n$.

Let $\tilde{\Omega}_\alpha = \{x \in \mathbb{R}^n : \frac{x_0}{r_\alpha} + x_\alpha \in \Omega\}$. Since Ω is smooth, it follows that the limit set $\tilde{\Omega}_\infty$ of $\tilde{\Omega}_\alpha$ as $\alpha \to +\infty$ is an open set. The next lemma in particular shows that $\tilde{\Omega}_\infty = \mathbb{R}^n$.

Lemma 2.4. *Up to a subsequence, $r_\alpha \to +\infty$ and $r_\alpha \operatorname{dist}(x_\alpha, \partial\Omega) \to +\infty$ as $\alpha \to +\infty$.*

Proof. Since $\tilde{v}_0$ is a nontrivial non-negative weak solution of (7), we claim that $r_\alpha \to +\infty$ as $\alpha \to +\infty$. Otherwise, there exists a constant $c > 0$ such that $r_\alpha \operatorname{dist}(x_\alpha, \partial\Omega) \le c$ for all α. In this case, after a suitable change of coordinates, we can assume that

$$\tilde{\Omega}_\infty = H = \{x \in \mathbb{R}^n : x_n > 0\}.$$

Using the fact that $\tilde{v}_0$ is a non-negative weak solution of (18), it follows by Step 6 that $\tilde{v}_0$ must vanish identically, which is a clear contradiction. $\square$

Now let

$$\eta \in C_0^\infty(\mathbb{R}^n), \qquad 0 \le \eta \le 1, \qquad \eta \equiv 1 \quad \text{in } B_1(0) \quad \text{and} \quad \eta = 0 \quad \text{outside } B_2(0)$$

and

$$w_\alpha(x) = v_\alpha(x) - r_\alpha^{\frac{n-p}{p}} \eta(\overline{r}_\alpha(x - x_\alpha)) \cdot \tilde{v}_0(r_\alpha(x - x_\alpha)) \in W_0^{1,p}(\Omega, \mathbb{R}^k),$$

where the sequence $(\overline{r}_\alpha)_\alpha$ is chosen in such a way that

$$\tilde{r}_\alpha = r_\alpha (\overline{r}_\alpha)^{-1} \to +\infty \quad \text{and} \quad \overline{r}_\alpha \operatorname{dist}(x_\alpha, \partial\Omega) \to +\infty.$$

Note that the maps v and $\hat{B}_\alpha$ presented in Step 7 are given by $\tilde{v}_0$ and $r_\alpha^{\frac{n-p}{p}} \tilde{v}_0(r_\alpha(x - x_\alpha))$, respectively.

The last lemma in the proof of Step 7 is as follows.

Lemma 2.5. *We have $w_\alpha \rightharpoonup 0$ in $W_0^{1,p}(\Omega, \mathbb{R}^k)$ and $D\mathcal{I}(w_\alpha) \to 0$ in $W_0^{1,p}(\Omega, \mathbb{R}^k)^*$ as $\alpha \to +\infty$. Moreover,*

$$\mathcal{I}(w_\alpha) = \mathcal{I}(v_\alpha) - \mathcal{J}(\tilde{v}_0) + o(1),$$

where $o(1) \to 0$ as $\alpha \to +\infty$.

Proof. Consider the rescaling

$$\tilde{w}_\alpha(x) = r_\alpha^{-\frac{n-p}{p}} w\left(\frac{x}{r_\alpha} + x_\alpha\right) = \tilde{v}_\alpha(x) - \tilde{v}_0(x)\eta\left(\frac{x}{\tilde{r}_\alpha}\right)$$

and let

$$\eta_\alpha(x) = \eta\left(\frac{x}{\tilde{r}_\alpha}\right).$$

We have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla(\eta_\alpha \tilde{v}_0 - \tilde{v}_0)|^p \, dx &= \int_{\mathbb{R}^n} |\nabla((\eta_\alpha - 1)\tilde{v}_0)|^p \, dx \\ &\le c \sum_{i=1}^k \int_{\mathbb{R}^n} |\nabla \tilde{v}_0^i|^p |(\eta_\alpha - 1)|^p \, dx + c \int_{\mathbb{R}^n} |\tilde{v}_0|^p |\nabla(\eta_\alpha - 1)|^p \, dx \\ &\le c \int_{\mathbb{R}^n \setminus B_{\tilde{r}_\alpha}(0)} |\nabla \tilde{v}_0|^p \, dx + c\, \tilde{r}_\alpha^{-p} \int_{B_{2\tilde{r}_\alpha}(0) \setminus B_{\tilde{r}_\alpha}(0)} |\tilde{v}_0|^p \, dx. \end{aligned}$$

Since $|\nabla \tilde{v}_0^i|^p$ is integrable on $\mathbb{R}^n$, the first term of the above inequality tends to 0 as $\tilde{r}_\alpha \to +\infty$. In addition, by Hölder's inequality and the fact that $|\tilde{v}_0^i|^{p^*}$ is integrable on $\mathbb{R}^n$, we conclude that the second term tends to 0 as $\alpha \to +\infty$. Thus, from what we just have proved, we derive

$$\tilde{w}_\alpha = \tilde{v}_\alpha - \tilde{v}_0 + o(1),$$

where $o(1) \to 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$.

However,

$$\mathcal{J}(\tilde{w}_\alpha) = \mathcal{I}(w_\alpha)$$

and by the Brézis–Lieb lemma we have

$$\mathcal{J}(\tilde{w}_\alpha) = \mathcal{J}(\tilde{v}_\alpha) - \mathcal{J}(\tilde{v}_0) + o(1) = \mathcal{I}(v_\alpha) - \mathcal{J}(\tilde{v}_0) + o(1),$$

where $o(1) \to 0$ as $\alpha \to +\infty$. Consequently,

$$\mathcal{I}(w_\alpha) = \mathcal{I}(v_\alpha) - \mathcal{J}(\tilde{v}_0) + o(1),$$

so that $(\mathcal{I}(w_\alpha))_\alpha$ is bounded. We conclude the proof by showing that $D\mathcal{I}(w_\alpha) \to 0$ in $W_0^{1,p}(\Omega, \mathbb{R}^k)^*$. In fact, since $(v_\alpha)_\alpha$ is a Palais–Smale sequence for $\mathcal{I}$ and $\tilde{v}_0$ is a critical point of $\mathcal{J}$, we obtain

$$\begin{aligned}\|D\mathcal{I}(w_\alpha)\| &= \|D\mathcal{J}(\tilde{w}_\alpha)\| \le \|D\mathcal{J}(\tilde{v}_\alpha)\| + \|D\mathcal{J}(\tilde{v}_0)\| + o(1)\\ &= \|D\mathcal{I}(v_\alpha)\| + o(1) = o(1). \quad \square \quad \square\end{aligned}$$

3. Proof of Theorem 1.2

In this last section, we characterize the existence of solutions of the potential system (7) generated by solutions of (4).

Proof of Theorem 1.2. Assume first that (7) admits a solution of the form tu, with $t = (t_1, \dots, t_k) \in \mathbb{R}^k \setminus \{0\}$, and u is a nontrivial solution of (4). Then, using the fact that F is even and p^*-homogeneous, we can easily check that $(-\Delta_p u)t^p = |u|^{p^*-2}u\nabla F(t)$. Since u is a nontrivial solution of (4), we obtain $\nabla F(t) = t^p$. Conversely, let t^p and $\nabla F(t)$ be parallel vectors, so that $\nabla F(t) = \theta t^p$ for some non-null number $\theta \in \mathbb{R}$. Taking the Euclidean inner product on both sides with the vector t, we obtain

$$p^* F(t) = \nabla F(t) \cdot t = \theta |t|_p^p,$$

so that θ is positive. Let $c > 0$ be such that $c^{p^*-p}\theta = 1$ and let u be a nontrivial solution of (4). We can easily deduce that the map $c \to u$ satisfies (7).

Suppose now that t^p and $\nabla F(t)$ are parallel and let t_0 be a vector parallel to t. Since F is even, arguing, if necessary, with $-t_0$ in the place of t_0, we can assume that t_0 and t point to the same direction. In particular, the same holds for the vectors t_0 and $\nabla F(t_0)$, so that $\nabla F(t_0) = \lambda t_0^p$ for some number $\lambda > 0$. Let u_0 be the unique radial solution of (4) satisfying $u_0(0) = 1$. Finally, the map $u = t_0 u_0$ is radial, clearly satisfies $u(0) = t_0$ and, by straightforward computation, solves (7). $\square$

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