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Computing the first eigenvalue of the p-Laplacian via the inverse power method

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Abstract

In this paper, we discuss a new method for computing the first Dirichlet eigenvalue of the p-Laplacian inspired by the inverse power method in finite dimensional linear algebra. The iterative technique is independent of the particular method used in solving the p-Laplacian equation and therefore can be made as efficient as the latter. The method is validated theoretically for any ball in $\mathbb{R}^n$ if $p > 1$ and for any bounded domain in the particular case $p = 2$. For $p > 2$ the method is validated numerically for the square.

Keywords: p-Laplacian; First eigenvalue; Comparison principle; Power method

1. Introduction

In finite dimensional linear algebra, the *power method* is often used in order to compute the first eigenvalue of invertible linear operators defined on Euclidean spaces (see [9], for instance). Briefly, given an invertible linear operator L, one picks a vector x and forms the sequence

$$x, Ax, A^2x, \ldots.$$

In order to produce this sequence, it is not necessary to get the powers of A explicitly, since each vector in the sequence can be obtained from the previous one by multiplying it by A. It is

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easy to show that the sequence converges, up to scaling, to the dominant eigenvector. Thus the largest eigenvalue can be found. In order to obtain the first (and therefore smallest) eigenvalue, one considers instead powers of A^{-1}, a method which is sometimes called the *inverse power method* or *inverse iteration*. More specifically, if $\{y_n\}_{n\in\mathbb{N}}$ denotes the normalized sequence of vectors produced by the inverse power method, then the first eigenvalue λ_1 of A can be explicitly given as (see [5])

$$\lambda_1 = \lim_{n\to\infty}\left(y_n \cdot A^{-1}y_n\right),$$

where $\cdot$ denotes the Euclidean inner product.

In this work we carry this idea further, to nonlinear operators in infinite dimensional spaces, in order to develop a numerical method to compute the first eigenvalue of the nonlinear degenerate elliptic p-Laplacian operator. In order to describe the technique, first we establish some notation and recall some well-known results. Throughout this paper, Ω will denote a smooth bounded region in $\mathbb{R}^N$, $N \geqslant 1$, and Δ_p will denote the p-Laplacian operator, that is,

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \tag{1}$$

for $1 < p < \infty$. We consider the Dirichlet eigenvalue problem for the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

As it is well known, the first eigenvalue $\lambda_p(\Omega)$ is positive, simple, can be variationally characterized, and the corresponding eigenfunction belongs to the Hölder space $C^{1,\alpha}(\overline{\Omega})$, does not change sign and therefore can be taken positive. If $p = 2$, when Δ_p becomes the Laplacian operator Δ, the value of $\lambda_p(\Omega)$ is well known for domains with simple geometry; for more general domains it can be determined by several methods. For $p > 1$ and $N = 1$, the value of $\lambda_p(\Omega)$ is known: if $\Omega = (a, b)$, then

$$\lambda_p(\Omega) = (p-1)\left(\frac{\pi_p}{b-a}\right)^p,$$

where

$$\pi_p := 2\int_0^1 \frac{ds}{\sqrt[p]{1-s^p}} = 2\frac{\pi/p}{\operatorname{sen}(\pi/p)}.$$

However, if $p \neq 2$ and $N \geqslant 2$, the value of $\lambda_p(\Omega)$ is not explicitly known, not even for simple domains such as a ball or a square. In such cases, there are few available numerical methods for finding $\lambda_p(\Omega)$. In the absence of an exact value or even a good approximation for $\lambda_p(\Omega)$, lower bounds play an important role in its estimation, being of special interest in the literature (upper bounds are more easily obtainable from the variational characterization of $\lambda_p(\Omega)$). An important lower bound for $\lambda_p(\Omega)$ (see [7]) is $\lambda_p(B)$ where $B \subset \mathbb{R}^N$ is a ball centered at the origin and with the same N-dimensional Lebesgue measure of Ω.

We propose a method based on the inverse power method for obtaining $\lambda_p(\Omega)$ and prove its applicability in the case when $\Omega = B$ (without loss of generality we choose B to be the unit ball).

We believe this specific result is very relevant and certainly will contribute to research on quasilinear problems in which spherical geometry or the estimation of $\lambda_p(\Omega)$ are important. Moreover, the main results that support the method are valid for a general domain (even unbounded) and thus they lead us to conjecture that our method is applicable to a more general class of domains.

In the remainder of this introduction we describe in more detail the main results of this paper and our conjecture.

Let $W_0^{1,p}(\Omega)$ denote the standard Sobolev space with norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$. We recall that $u \in W_0^{1,p}(\Omega)$ is a *weak solution* to the homogeneous Dirichlet problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

for a given function $f \in L^{p'}(\Omega)$, where $p' = p/(p-1)$ denotes the conjugate exponent of p, if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \tag{4}$$

for every test function $v \in W_0^{1,p}(\Omega)$. It is well known that if $f \in C^0(\overline{\Omega})$ then the corresponding weak solution u belongs to $C^{1,\alpha}(\overline{\Omega})$ and the inverse operator $(-\Delta_p)^{-1} : C^0(\overline{\Omega}) \to W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^0(\overline{\Omega})$ is continuous and compact.

The technique runs as follows. First, iteratively define a sequence of functions $(\phi_n)_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ by setting $\phi_0 \equiv 1$ and, for $n = 1, 2, 3, \ldots$, letting ϕ_n be the solution to the Dirichlet problem

$$\begin{cases} -\Delta_p \phi_n = \phi_{n-1}^{p-1} & \text{in } \Omega, \\ \phi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Then, for $n \geqslant 1$, define the following sequences of real numbers

$$\gamma_n := \inf_\Omega \left(\frac{\phi_n}{\phi_{n+1}} \right)^{p-1} \tag{6}$$

and

$$\Gamma_n := \sup_\Omega \frac{\phi_n}{\phi_{n+1}} = \left\| \frac{\phi_n}{\phi_{n+1}} \right\|_{L^\infty(\Omega)}. \tag{7}$$

We show that sequence (6) is indeed well defined and bounded above by the first Dirichlet eigenvalue of the p-Laplacian λ_p. This result in itself is of particular importance, since lower bounds for λ_p are hard to obtain. For sequence (7), we give strong evidence that it is well defined for at least all sufficiently large n (we prove this is true for balls and for general domains in the special case $p = 2$) and that it is bounded below by λ_p. We also show that sequence (6) is increasing (which implies among other things that we can obtain successively better lower bounds for λ_p), whereas sequence (7) is decreasing. Thus, the limit

$$\gamma := \lim \gamma_n$$

exists and is finite, the limit

$$\Gamma := \lim \Gamma_n$$

exists and is likely finite (it is definitely finite in some cases), and they satisfy

$$\gamma \leqslant \lambda_p \leqslant \Gamma.$$

We conjecture that

$$\lambda_p = \gamma = \Gamma. \tag{8}$$

If this conjecture is true, we also show that the first eigenfunction might be construed as the limit of a certain scaling of sequence (ϕ_n). We are able to show that this indeed happens in the special cases of balls, for general p, and general domains, for $p = 2$. In the latter case we use the Hilbert space structure of the space $W_0^{1,2}(\Omega)$ and some well-known results for generalized Fourier expansions in eigenfunctions of the Laplacian. Needless to say, such an argument does not work for $p \neq 2$. However, numerical experiments indicate that for general p we do have

$$\gamma = \Gamma$$

and that these values approach known values of λ_p obtained by other authors using different techniques.

We also consider the sequence of numbers

$$\nu_n = \left(\frac{\|\phi_n\|_{L^p(\Omega)}}{\|\phi_{n+1}\|_{L^p(\Omega)}} \right)^{p-1} \tag{9}$$

and show that it is bounded below by λ_p and above by the sequence (Γ_n), so that it is also our (independent) conjecture that

$$\lambda_p = \nu := \lim \nu_n. \tag{10}$$

In numerical experiments, it is observed that the convergence of this sequence is significantly faster than the convergences of the above two sequences.

This paper is organized as follows. In Section 2, we prove the monotonicity of sequences (γ_n) and (Γ_n), and that the first eigenvalue is an upper bound for sequence (γ_n). We also study sequence (ν_n) and find that the first eigenvalue is a lower bound for this sequence as well as for sequence (Γ_n). In Section 3, we show that if conjecture (8) is true, a certain scaled limit of sequence (ϕ_n) might approach the first eigenfunction. Section 4 gives a complete proof of conjecture (8) in the case of n-dimensional balls. In Section 5, we prove the convergence of all three sequences to the first eigenvalue in the case $p = 2$. Section 6 shows that in principle this technique can be extended to yield higher eigenvalues and corresponding eigenfunctions, at least in the $p = 2$ case. Finally, in Section 7 we describe the numerical experiments performed and their results.

2. Behavior of sequences (γ_n), (Γ_n) and (v_n)

In the following, we denote by $\|u\|_\infty$ and $\|u\|_p$ respectively the L^∞ and L^p norms of a function u in Ω. Let $(\phi_n)_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \cap C^{1.\alpha}(\overline{\Omega})$ be the sequence defined by $\phi_0 \equiv 1$ and, for $n \geqslant 1$, ϕ_n is the solution to the Dirichlet problem

$$\begin{cases} -\Delta_p \phi_n = \phi_{n-1}^{p-1} & \text{in } \Omega, \\ \phi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

2.1. Sequence (γ_n)

Set

$$\gamma_n := \inf_{\Omega} \left(\frac{\phi_n}{\phi_{n+1}} \right)^{p-1}.$$

First we show that sequence (γ_n) is well defined. In the sequel we will resort several times to the well-known comparison principle for the p-Laplacian (see [4], for instance). For easiness of reference we state it here.

Proposition 2.1 *(Comparison principle). Let $u_1, u_2 \in C^{1.\alpha}(\overline{\Omega})$ satisfy*

$$\begin{cases} -\Delta_p u_1 \leqslant -\Delta_p u_2 & \text{in } \Omega, \\ u_1 \leqslant u_2 & \text{on } \partial\Omega. \end{cases}$$

Then, $u_1 \leqslant u_2$ in Ω.

Proposition 2.2. *The sequence $(\phi_n)_{n\in\mathbb{N}}$ satisfies*

$$0 < \phi_n \leqslant \|\phi_1\|_\infty \phi_{n-1} \quad \text{in } \Omega$$

for every $n \geqslant 1$.

Proof. First we show that ϕ_n is positive by using a simple argument involving the comparison principle. The comparison principle already implies that $\phi_n \geqslant 0$.

Fix an arbitrary $x_0 \in \Omega$ and let $R > 0$ be such that the ball $B_R(x_0)$ centered at x_0 with radius R is contained in Ω. We will show that $\phi_n(x) > 0$ for all $x \in B_R(x_0)$.

Let ψ_1 be the solution to the Dirichlet problem

$$\begin{cases} -\Delta_p \psi_1 = 1 & \text{in } B_R(x_0), \\ \psi_1 = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

It is well known that the solution to this problem is radially symmetric, that is, $\psi_1(x) = v_1(r)$ for $r = |x - x_0|$, where v_1 satisfies the problem

$$\begin{cases} -\left(r^{N-1}|v_1'|^{p-2}v_1'\right)' = r^{N-1} & \text{for } 0 < r < R, \\ v_1'(0) = v_1(R) = 0. \end{cases}$$

Integrating the differential equation we find

$$v_1(r) = \int_r^R \left(\int_0^r \left(\frac{s}{r}\right)^{N-1} ds \right)^{\frac{1}{p-1}} dr$$

and hence $v_1 > 0$ for $0 < r < R$. Thus, since

$$\begin{cases} -\Delta_p \phi_1 = -\Delta_p \psi_1 & \text{in } B_R(x_0), \\ \phi_1 \geqslant 0 = \psi_1 & \text{on } \partial B_R(x_0), \end{cases}$$

it follows from the comparison principle that

$$\phi_1 \geqslant \psi_1 > 0 \quad \text{in } B_R(x_0).$$

For $n \geqslant 2$ define the sequence (ψ_n) iteratively by

$$\begin{cases} -\Delta_p \psi_{n+1} = \psi_n^{p-1} & \text{in } B_R(x_0), \\ \psi_{n+1} = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

As before, $\psi_{n+1}(x) = v_{n+1}(r)$ where v_{n+1} is the solution to the (nonlinear) problem

$$\begin{cases} -\left(r^{N-1} \left|v'_{n+1}\right|^{p-2} v'_{n+1}\right)' = r^{N-1} v_n^{p-1} & \text{for } 0 < r < R, \\ v'_{n+1}(0) = v_{n+1}(R) = 0, \end{cases}$$

so that

$$\psi_{n+1}(x) = \int_{|x-x_0|}^R \left(\int_0^r \left(\frac{s}{r}\right)^{N-1} v_n^{p-1}(s)\, ds \right)^{\frac{1}{p-1}} dr > 0 \quad \text{for } x \in B_R(x_0).$$

Assuming by induction that $\phi_n \geqslant \psi_n$ in $B_R(x_0)$, it follows that

$$\begin{cases} -\Delta_p \phi_{n+1} = \phi_n^{p-1} \geqslant \psi_n^{p-1} = -\Delta_p \psi_{n+1} & \text{in } B_R(x_0), \\ \phi_{n+1} \geqslant 0 = \psi_{n+1} & \text{on } \partial B_R(x_0), \end{cases}$$

which implies that $\phi_{n+1} \geqslant \psi_{n+1}$ in $B_R(x_0)$. Therefore, we conclude that $\phi_{n+1} \geqslant \psi_{n+1} > 0$ in $B_R(x_0)$ for all n. Since x_0 is arbitrary, it follows that $\phi_n > 0$ in Ω for all n.

It remains to show that $\phi_n \leqslant \|\phi_1\|_\infty \phi_{n-1}$. Again we use an induction argument together with the comparison principle. Trivially,

$$\phi_1 \leqslant \|\phi_1\|_\infty = \|\phi_1\|_\infty \phi_0.$$

Assume

$$\phi_n \leqslant \|\phi_1\| \phi_{n-1}.$$

We have

$$\begin{cases} -\Delta_p \phi_{n+1} = \phi_n^{p-1} \leqslant \|\phi_1\|_\infty^{p-1} \phi_{n-1}^{p-1} = -\Delta_p\big(\|\phi_1\|_\infty \phi_n\big) & \text{in } \Omega, \\ \phi_{n+1} = 0 = \|\phi_1\|_\infty \phi_n & \text{on } \partial\Omega, \end{cases}$$

whence

$$\phi_{n+1} \leqslant \|\phi_1\|_\infty \phi_n. \qquad \square$$

As a consequence of Proposition 2.2, it follows that

$$\|\phi_{n+1}\|_\infty \leqslant \|\phi_1\|_\infty \|\phi_n\|_\infty$$

and

$$\frac{\phi_n}{\phi_{n+1}} \geqslant \frac{1}{\|\phi_1\|_\infty} > 0.$$

Therefore, the sequence

$$\gamma_n = \inf_\Omega \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} \tag{12}$$

is well defined. Observe that

$$\gamma_0 = \inf_\Omega \left(\frac{1}{\phi_1}\right)^{p-1} = \frac{1}{\|\phi_1\|_\infty^{p-1}}. \tag{13}$$

Next we show that (γ_n) is an increasing sequence, bounded above by the first eigenvalue. We will need the following lemma:

Lemma 2.3. *Let $\Omega \subset R^N$ be a smooth bounded region and $h \in C^0(\overline{\Omega})$ a nonnegative function. If $u \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ is a positive solution to the Dirichlet problem*

$$\begin{cases} -\Delta_p u = \lambda_p u^{p-1} + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

then $h \equiv 0$ in Ω and consequently u is a positive eigenfunction corresponding to the first eigenvalue λ_p.

Proof. This proof is adapted from [1, Theorem 2.4] and based on the inequality

$$|\nabla w|^p \geqslant |\nabla u|^{p-2} \nabla u \cdot \nabla\left(\frac{w^p}{u^{p-1}}\right), \tag{15}$$

valid for all differentiable functions u, w in Ω that satisfy $u > 0$ and $w \geqslant 0$; this inequality follows from Picone's identity.

Multiplying Eq. (14) by any $v \in W_0^{1,p}(\Omega)$ and integrating on Ω, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx = \int_\Omega \big(\lambda_p u^{p-1} + h\big)v\,dx. \tag{16}$$

Let $u_p \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ be a positive eigenfunction corresponding to λ_p. Applying (15) with $w = u_p$ and integrating on Ω, we find that

$$\int_\Omega |\nabla u_p|^p\,dx \geqslant \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\left(\frac{u_p^p}{u^{p-1}}\right)dx.$$

Now, since $u > 0$ in Ω, by Hopf's lemma we have $u_p^p/u^{p-1} \in W_0^{1,p}(\Omega)$. Therefore, we can apply (16) to conclude that

$$\int_\Omega |\nabla u_p|^p\,dx \geqslant \int_\Omega \big(\lambda_p u^{p-1} + h\big)\frac{u_p^p}{u^{p-1}}\,dx.$$

Thus,

$$0 = \int_\Omega |\nabla u_p|^p\,dx - \int_\Omega \lambda_p u_p^p\,dx \geqslant \int_\Omega h\frac{\phi^p}{u^{p-1}}\,dx \geqslant 0,$$

which implies that $h \equiv 0$. $\square$

Proposition 2.4. *For all $n \geqslant 2$ the following hold*:

(i) $\gamma_0 < \lambda_p$.
(ii) $\gamma_0 \leqslant \gamma_n \leqslant \gamma_{n+1} < \lambda_p$.
(iii) *There exists*

$$\gamma := \lim \gamma_n$$

and $\gamma_0 \leqslant \gamma \leqslant \lambda_p$.

Proof. Property (iii) is a direct consequence of (i) and (ii). We prove (i) by a contradiction argument. Assume $\gamma_0 \geqslant \lambda_p$. Then, setting

$$h = 1 - \lambda_p \phi_1^{p-1},$$

it follows by (13) that

$$h \geqslant 1 - \gamma_0 \phi_1^{p-1} \geqslant 0.$$

Write

$$\begin{cases} -\Delta_p \phi_1 = 1 = \lambda_p \phi_1^{p-1} + h & \text{in } \Omega, \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

From Lemma 2.3, we conclude that $h \equiv 0$ and so $\lambda_p \phi_1 \equiv 1$. This implies that ϕ_1 is constant, contradicting $-\Delta_p \phi_1 = 1$ in Ω.

In order to prove (ii), observe that $\gamma_0 \leqslant \gamma_n$ follows immediately from Proposition 2.2:

$$\gamma_0 = \frac{1}{\|\phi_1\|_\infty^{p-1}} \leqslant \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1}.$$

The monotonicity of the sequence (γ_n) can be shown by using the comparison principle. We have by definition

$$\begin{cases} -\Delta_p \phi_n = \phi_{n-1}^{p-1} \geqslant \gamma_{n-1} \phi_n^{p-1} = -\Delta_p \big(\gamma_{n-1}^{1/(p-1)} \phi_{n+1}\big) & \text{in } \Omega, \\ \phi_n = 0 = \phi_{n+1}, & \text{on } \partial\Omega, \end{cases}$$

whence

$$\phi_n \geqslant \gamma_{n-1}^{1/(p-1)} \phi_{n+1},$$

and, therefore

$$\gamma_n = \inf_\Omega \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} \geqslant \gamma_{n-1}.$$

Finally, in order to verify that $\gamma_n < \lambda_p$ we use again a contradiction argument. Suppose that $\gamma_n \geqslant \lambda_p$ for some n. Then,

$$\lambda_p \phi_{n+1}^{p-1} \leqslant \gamma_n \phi_{n+1}^{p-1} \leqslant \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} \phi_{n+1}^{p-1} = \phi_n^{p-1},$$

the second inequality a consequence of (12). Thus,

$$h := \phi_n^{p-1} - \lambda_p \phi_{n+1}^{p-1} \geqslant 0$$

in Ω. Since

$$\begin{cases} -\Delta_p \phi_{n+1} = \phi_n^{p-1} = \lambda_p \phi_{n+1}^{p-1} + h & \text{in } \Omega, \\ \phi_{n+1} = 0 & \text{on } \partial\Omega, \end{cases}$$

it follows from Lemma 2.3 that $h \equiv 0$. Thus,

$$\phi_n^{p-1} = \lambda_p \phi_{n+1}^{p-1} \tag{17}$$

and hence

$$\lambda_p \equiv \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} = \inf_\Omega \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} = \gamma_n.$$

Furthermore, it also follows from (17) that

$$\phi_{n-1}^{p-1} = -\Delta_p \phi_n = -\Delta_p\big(\lambda_p^{1/(p-1)}\phi_{n+1}\big) = \lambda_p(-\Delta_p\phi_{n+1}) = \lambda_p \phi_n^{p-1}$$

whence

$$\lambda_p \equiv \left(\frac{\phi_{n-1}}{\phi_n}\right)^{p-1} = \inf_\Omega \left(\frac{\phi_{n-1}}{\phi_n}\right)^{p-1} = \gamma_{n-1}.$$

Proceeding recursively we obtain

$$\lambda_p = \gamma_0,$$

which contradicts (i). $\square$

As a consequence from Propositions 2.2 and 2.4(i), we obtain the following behavior for the sequence (ϕ_n):

Corollary 2.5. $\frac{\phi_n}{\|\phi_1\|_\infty^n} \to 0$ *monotonically and uniformly.*

Proof. Set

$$w_n := \frac{\phi_n}{\|\phi_1\|_\infty^n},$$

so that, by (13),

$$\begin{cases} -\Delta_p w_n = \gamma_0 w_{n-1}^{p-1} & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from Proposition 2.2 that the sequence (w_n) is decreasing and uniformly bounded, since

$$\frac{w_{n+1}}{w_n} = \frac{\|\phi_1\|_\infty^n}{\|\phi_1\|_\infty^{n+1}} \frac{\phi_{n+1}}{\phi_n} = \frac{1}{\|\phi_1\|_\infty} \frac{\phi_{n+1}}{\phi_n} \leqslant 1$$

and

$$\|w_n\|_\infty = \frac{\|\phi_n\|_\infty}{\|\phi_1\|_\infty^n} \leqslant \frac{\|\phi_{n-1}\|_\infty \|\phi_1\|_\infty}{\|\phi_1\|_\infty^n} \leqslant \frac{\|\phi_{n-2}\|_\infty \|\phi_1\|_\infty^2}{\|\phi_1\|_\infty^n} \leqslant \cdots \leqslant 1.$$

The uniform boundedness of (w_n) together with the compacity of the operator $(-\Delta_p)^{-1}: C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ imply that $(w_n) = (-\Delta_p)^{-1}(\gamma_0 w_{n-1})$ has a subsequence which converges uniformly to some function $w \in C^0(\overline{\Omega})$. The monotonicity of (w_n) guarantees that the whole sequence converges uniformly and monotonically to w. The continuity of $-\Delta_p^{-1}$ implies that

$w = (-\Delta_p)^{-1}(\gamma_0 w)$. Therefore,

$$\begin{cases} -\Delta_p w = \gamma_0 w^{p-1} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\gamma_0 < \lambda_p$ and λ_p is the first eigenvalue for $-\Delta_p$, we conclude that $w = 0$. $\square$

2.2. *Sequence* (Γ_n)

Set

$$\Gamma_n := \sup_{\Omega} \left(\frac{\phi_n}{\phi_{n+1}} \right)^{p-1} = \left\| \frac{\phi_n}{\phi_{n+1}} \right\|_{L^\infty(\Omega)}^{p-1}. \tag{18}$$

Observe that

$$\Gamma_0 = \left\| \frac{\phi_0}{\phi_1} \right\|_\infty^{p-1} = \infty,$$

since $\phi_0 = 1$ on $\overline{\Omega}$ and $\phi_1 = 0$ on the boundary $\partial\Omega$. However, if one can guarantee that Γ_{n_0} is finite for some n_0, then sequence (Γ_n) is well defined from n_0 on:

Proposition 2.6. *Assume $\Gamma_{n_0} < \infty$ for some $n_0 \geqslant 1$. Then*

$$\left(\frac{\phi_n}{\phi_{n+1}} \right)^{p-1} \leqslant \Gamma_{n_0} \quad \text{for all } n \geqslant n_0$$

and therefore (Γ_n) is well defined and bounded from above for $n \geqslant n_0$. Moreover, the sequence (Γ_n) is decreasing for $n \geqslant n_0$.

Proof. Proceeding by induction, assume we have shown $\Gamma_{n_0+k} \leqslant \cdots \leqslant \Gamma_{n_0}$. Then, for $j = n_0 + k$ we observe that

$$-\Delta_p \phi_{j+1} = \phi_j^{p-1} = \left(\frac{\phi_j}{\phi_{j+1}} \right)^{p-1} \phi_{j+1}^{p-1} \leqslant \Gamma_j \phi_{j+1}^{p-1} = -\Delta_p \big(\Gamma_j^{\frac{1}{p-1}} \phi_{j+2} \big)$$

in Ω, and

$$\phi_{j+1} = 0 = \Gamma_j^{\frac{1}{p-1}} \phi_{j+2}$$

on $\partial\Omega$. Thus, it follows from the comparison principle that

$$\phi_{j+1} \leqslant \Gamma_j^{\frac{1}{p-1}} \phi_{j+2} \quad \text{in } \Omega.$$

Hence,

$$\Gamma_{j+1} = \left\| \frac{\phi_{j+1}}{\phi_{j+2}} \right\|_\infty^{p-1} \leqslant \Gamma_j. \qquad \square$$

For special domains, we are able to prove that there exists n_0 such that Γ_{n_0} is finite. Indeed, the following result holds:

Proposition 2.7. *Let $\Omega = B_R$ be the ball of center at the origin and radius R. Then Γ_1 is finite.*

Proof. Using the same notation as in Proposition 2.2, we have $\phi_n(x) = v_n(r)$ with

$$v_n(r) = \int_r^1 \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} v_{n-1}(s)^{p-1}\, ds \right)^{\frac{1}{p-1}} dr. \tag{19}$$

Thus, if $x \in \partial B_R$, L'Hôpital's rule implies that

$$\frac{\phi_1(x)}{\phi_2(x)} = \lim_{r\to R^-} \frac{v_1(r)}{v_2(r)} = \lim_{r\to R^-} \frac{v_1'(r)}{v_2'(r)} = \left(\frac{\int_0^R s^{N-1}\, ds}{\int_0^R s^{N-1} v_1(s)^{p-1}\, ds} \right)^{\frac{1}{p-1}} < \infty. \qquad \square$$

2.3. Sequence (ν_n)

Set

$$\nu_n = \left(\frac{\|\phi_n\|_{L^p(\Omega)}}{\|\phi_{n+1}\|_{L^p(\Omega)}} \right)^{p-1}.$$

Clearly, sequence (ν_n) is well defined. We show that both it and sequence (Γ_n) are bounded below by the first eigenvalue.

Proposition 2.8. *There holds*

$$\lambda_p \leqslant \nu_n \leqslant \Gamma_n$$

for all $n \geqslant 1$.

Proof. By (4) and Hölder's inequality we have

$$\|\nabla\phi_{n+1}\|_p^p = \int_\Omega |\nabla\phi_{n+1}|^p\, dx = \int_\Omega \phi_n^{p-1}\phi_{n+1}\, dx \leqslant \left\|\phi_n^{p-1}\right\|_{p'} \|\phi_{n+1}\|_p,$$

whence

$$\|\nabla\phi_{n+1}\|_p^p \leqslant \|\phi_n\|_p^{p-1} \|\phi_{n+1}\|_p.$$

Hence, from this and the variational characterization of the first eigenvalue

$$\lambda_p = \inf_{v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_p}{\|v\|_p},$$

it follows that

$$
\begin{aligned}
\lambda_p &\leqslant \frac{\|\nabla\phi_{n+1}\|_p^p}{\|\phi_{n+1}\|_p^p} \leqslant \frac{\|\phi_n\|_p^{p-1}\|\phi_{n+1}\|_p}{\|\phi_{n+1}\|_p^p} \\
&= \nu_n \\
&= \frac{1}{\|\phi_{n+1}\|_p^{p-1}}\left(\int_\Omega \left(\frac{\phi_n}{\phi_{n+1}}\right)^p \phi_{n+1}^p\,dx\right)^{\frac{p-1}{p}} \\
&\leqslant \frac{1}{\|\phi_{n+1}\|_p^{p-1}}\left\|\frac{\phi_n}{\phi_{n+1}}\right\|_\infty^{p-1}\left(\int_\Omega \phi_{n+1}^p\,dx\right)^{\frac{p-1}{p}} \\
&= \Gamma_n. \qquad \square
\end{aligned}
$$

Corollary 2.9. *If*

$$\lim \Gamma_n = \lambda_p$$

then

$$\lim \nu_n = \lambda_p.$$

As an interesting consequence of Propositions 2.4 and 2.8 we have

Corollary 2.10. *If Ω is connected and $\Gamma_{n_0} < \infty$ for some $n_0 \geqslant 1$, then for each $n \geqslant n_0$ there exists at least one $x_n \in \overline{\Omega}$ such that*

$$\lambda_p = \frac{\phi_n(x_n)}{\phi_{n+1}(x_n)}.$$

3. The first eigenvalue and the first eigenfunction

Recall that if

$$\gamma = \lim \gamma_n,$$

then it follows from Proposition 2.4 that

$$\gamma \leqslant \lambda_p.$$

If we set

$$\Gamma := \lim \Gamma_n$$

and

$$\nu := \lim \nu_n,$$

we have from Propositions 2.6 and 2.8 that $\nu < \infty$ and

$$\lambda_p \leqslant \nu \leqslant \Gamma.$$

We conjecture the following:

Conjecture 3.1. *There holds*

$$\lambda_p = \gamma = \Gamma = \nu. \tag{20}$$

In order to find a sequence of functions approximating the first eigenfunction, we define for each $n \in \mathbb{N}$ the function

$$u_n := \frac{\phi_n}{a_n},$$

where a_n is such that

$$\frac{a_n}{a_{n+1}} = \gamma_n^{\frac{1}{p-1}} = \inf_{\Omega} \frac{\phi_n}{\phi_{n+1}}.$$

For instance, if we set

$$a_1 := \|\phi_1\|_\infty,$$

then

$$a_2 = \frac{a_1}{\gamma_1^{1/(p-1)}} = \frac{a_1}{\inf_\Omega \frac{\phi_1}{\phi_2}} = \|\phi_1\|_\infty \left\| \frac{\phi_2}{\phi_1} \right\|_\infty$$

and, in general,

$$a_n = \frac{\|\phi_1\|_\infty}{\inf_\Omega \frac{\phi_1}{\phi_2}} \frac{1}{\inf_\Omega \frac{\phi_2}{\phi_3}} \cdots \frac{1}{\inf_\Omega \frac{\phi_{n-1}}{\phi_n}} = \|\phi_1\|_\infty \left\| \frac{\phi_2}{\phi_1} \right\|_\infty \left\| \frac{\phi_3}{\phi_2} \right\|_\infty \cdots \left\| \frac{\phi_n}{\phi_{n-1}} \right\|_\infty. \tag{21}$$

Since

$$\frac{\|\phi_k\|_\infty}{\|\phi_{k-1}\|_\infty} \leqslant \left\| \frac{\phi_k}{\phi_{k-1}} \right\|_\infty,$$

we have

$$a_n \geqslant \|\phi_1\|_\infty \frac{\|\phi_2\|_\infty}{\|\phi_1\|_\infty} \frac{\|\phi_3\|_\infty}{\|\phi_2\|_\infty} \cdots \frac{\|\phi_n\|_\infty}{\|\phi_{n-1}\|_\infty} = \|\phi_n\|_\infty.$$

Therefore,

$$u_n \leqslant \frac{\phi_n}{\|\phi_n\|_\infty} \leqslant 1.$$

Proposition 3.2. *Let $(u_n) \subset W_0^{1,p}(\Omega) \cap C^{1.\alpha}(\overline{\Omega})$ be the sequence of functions defined above. Then (u_n) is decreasing and satisfies*

$$\begin{cases} -\Delta_p u_{n+1} = \gamma_n u_n^{p-1} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, (u_n) converges uniformly to a function $u \in C^{1.\alpha}(\overline{\Omega})$ which satisfies

$$\begin{cases} -\Delta_p u = \gamma u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. We have

$$-\Delta_p u_{n+1} = \left(\frac{\phi_n}{a_{n+1}}\right)^{p-1} = \left(\frac{a_n}{a_{n+1}}\right)^{p-1} u_n^{p-1} = \gamma_n u_n^{p-1}.$$

Moreover,

$$u_{n+1} = \frac{\phi_{n+1}}{a_{n+1}} = \frac{\phi_{n+1}}{a_n} \inf_{\Omega} \frac{\phi_n}{\phi_{n+1}} \leqslant \frac{\phi_{n+1}}{a_n} \frac{\phi_n}{\phi_{n+1}} = \frac{\phi_n}{a_n} = u_n,$$

which proves that (u_n) is decreasing and therefore we can define a function u in $\overline{\Omega}$ by

$$u(x) := \lim u_n(x)$$

for each $x \in \overline{\Omega}$. Since $(u_n) \subset C^{1,\alpha}(\overline{\Omega})$, $0 \leqslant u_n \leqslant u_1$, and the operator $(-\Delta_p)^{-1} : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is compact, the whole sequence (u_n) converges to u uniformly and we can pass the limit in

$$-\Delta_p u_{n+1} = \gamma_n u_n^{p-1}$$

to obtain

$$-\Delta_p u = \gamma u^{p-1}. \qquad \square$$

In view of Conjecture 3.1, this result suggests that u is the eigenfunction corresponding to the first eigenvalue. However, since we are not able to guarantee that u is not the null function, this does not follow automatically. On the other hand, since Proposition 3.2 is independent of the conjecture, its proof would show that $\gamma = \lambda_p$.

We end this section by remarking that all results proved above are valid if we consider a positive weight $\omega(x)$ multiplying the right-hand side of the eigenvalue equation, that is, if we consider the equation $-\Delta_p u = \lambda\omega(x)|u|^{p-1}u$ in Ω. The above arguments are easily adapted to contemplate this case and the most remarkable change appears in the sequence (ν_n) which becomes

$$\nu_n = \left(\frac{\|\omega(x)^{1/p}\phi_n\|_p}{\|\omega(x)^{1/p}\phi_{n+1}\|_p}\right)^{p-1}.$$

4. Spherical domains

In this section we show the validity of Conjecture 3.1 for balls. Let $B = B_1(0) \subset \mathbb{R}^N$, $N \geqslant 2$, denote the unit ball centered at the origin. The following lemma in the form it is given was first stated in [2], even though it has already often been used as a technical tool in differential geometry. A proof is provided for completeness.

Lemma 4.1. *Let $f, g : [a,b] \to R$ be continuous on $[a,b]$ and differentiable in (a,b). Suppose $g'(x) \neq 0$ for all $x \in (a,b)$. If $\frac{f'}{g'}$ is (strictly) increasing [decreasing], then both $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are (strictly) increasing [decreasing].*

Proof. Assume $\frac{f'}{g'}$ is increasing. Then the Cauchy mean value theorem implies that for each $x \in (a,b)$ there exists $y \in (a,x)$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(y)}{g'(y)} \leqslant \frac{f'(x)}{g'(x)}.$$

On the other hand, since $g' \neq 0$ we always have

$$\frac{g'(x)}{g(x)-g(a)} > 0.$$

Thus,

$$\begin{aligned} \frac{d}{dx}\left(\frac{f(x)-f(a)}{g(x)-g(a)}\right) &= \frac{f'(x)}{g(x)-g(a)} - \frac{g'(x)}{g(x)-g(a)}\frac{f(x)-f(a)}{g(x)-g(a)} \\ &\geqslant \frac{f'(x)}{g(x)-g(a)} - \frac{g'(x)}{g(x)-g(a)}\frac{f'(x)}{g'(x)} = 0, \end{aligned}$$

and so $\frac{f(x)-f(a)}{g(x)-g(a)}$ is increasing.

If $\frac{f'}{g'}$ is decreasing, then the same arguments prove that $\frac{f(x)-f(a)}{g(x)-g(a)}$ is decreasing. Moreover, the above inequalities are strict if the monotonicity of $\frac{f'}{g'}$ is strict. The proof for $\frac{f(x)-f(b)}{g(x)-g(b)}$ is similar. □

We use this lemma in order to show that for each $n \geqslant 0$ the quotient $\frac{\phi_n}{\phi_{n+1}}$ is increasing as a function of $r = |x|$:

Theorem 4.2. *Let $p > 1$ and for $r \in [0,1]$ set*

$$\phi_0(r) \equiv 1,$$

$$\phi_n(r) = \int_r^1 \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \phi_{n-1}^{p-1}(s)\,ds\right)^{\frac{1}{p-1}} d\theta, \quad \text{if } n \geqslant 1.$$

Then, for each $n \geqslant 1$ *the function* ϕ_n *is strictly decreasing and for each* $n \geqslant 0$ *the quotient* $\frac{\phi_n}{\phi_{n+1}}$ *is strictly increasing on* $[0, 1]$.

Proof. As

$$\phi_n'(r) = -\left(\int_0^r \left(\frac{s}{r}\right)^{N-1} \phi_{n-1}^{p-1}(s)\, ds\right)^{\frac{1}{p-1}} < 0,$$

for $r > 0$, the functions ϕ_n are strictly decreasing for $n \geqslant 1$.

In particular, the quotient $\frac{\phi_n}{\phi_{n+1}}$ is strictly increasing when $n = 0$. In order to show that the quotients are strictly increasing for $n \geqslant 1$, we use an induction argument. Assume that the quotient $\frac{\phi_{n-1}}{\phi_n}$ is strictly increasing for some $n \geqslant 1$. Noticing that $\phi_n(1) = \phi_{n+1}(1) = 0$, we can write

$$\frac{\phi_n}{\phi_{n+1}}(r) = \frac{\phi_n(r) - \phi_n(1)}{\phi_{n+1}(r) - \phi_{n+1}(1)}.$$

We will apply the previous lemma in order to show that the quotient in the right-hand side of this equation is strictly increasing. For this, it is enough to verify that

$$\frac{\phi_n'(r)}{\phi_{n+1}'(r)} = \left(\frac{\int_0^r s^{N-1}\phi_{n-1}^{p-1}(s)\, ds}{\int_0^r s^{N-1}\phi_n^{p-1}(s)\, ds}\right)^{\frac{1}{p-1}}$$

is increasing. Since the map $\xi \mapsto \xi^{\frac{1}{p-1}}$ is increasing, this is equivalent to showing that

$$\frac{\int_0^r s^{N-1}\phi_{n-1}^{p-1}(s)\, ds}{\int_0^r s^{N-1}\phi_n^{p-1}(s)\, ds}$$

is increasing. But this is itself a consequence of the lemma, for both $\int_0^r s^{N-1}\phi_{n-1}^{p-1}(s)\, ds$ and $\int_0^r s^{N-1}\phi_n^{p-1}(s)\, ds$ equal zero at $r = 0$ and

$$\frac{(\int_0^r s^{N-1}\phi_{n-1}^{p-1}(s)\, ds)'}{(\int_0^r s^{N-1}\phi_n^{p-1}(s)\, ds)'} = \left(\frac{\phi_{n-1}(r)}{\phi_n(r)}\right)^{p-1},$$

which is strictly increasing by the induction hypothesis. $\square$

Corollary 4.3.

$$\gamma_n = \inf_B \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} = \left(\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}\right)^{p-1}.$$

Proof. Since ϕ_n and ϕ_{n+1} are decreasing functions, we have $\|\phi_n\|_\infty = \phi_n(0)$ and $\|\phi_{n+1}\|_\infty = \phi_{n+1}(0)$. Therefore, since $\frac{\phi_n}{\phi_{n+1}}$ is increasing,

$$\inf_B \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} = \left(\frac{\phi_n(0)}{\phi_{n+1}(0)}\right)^{p-1} = \left(\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}\right)^{p-1}. \qquad \square$$

Theorem 4.4. *Let (u_n) be the sequence defined by*

$$u_n := \frac{\phi_n}{\|\phi_n\|}$$

for each $n \in \mathbb{N}$. Then

$$\gamma = \lambda_p(B)$$

and (u_n) converges uniformly (and monotonically) to a positive function $u \in C^{1,\alpha}(B)$ such that $\|u\|_\infty = 1$ and

$$\begin{cases} -\Delta_p u = \lambda_p u^{p-1} & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{22}$$

Proof. Notice that sequence (u_n) is the same as that defined in Proposition 3.2, since a_n defined in (21) is now, in view of Corollary 4.3,

$$a_n = \frac{\|\phi_1\|_\infty}{\inf_\Omega \frac{\phi_1}{\phi_2}} \frac{1}{\inf_\Omega \frac{\phi_2}{\phi_3}} \cdots \frac{1}{\inf_\Omega \frac{\phi_{n-1}}{\phi_n}} = \frac{\|\phi_1\|_\infty}{\frac{\|\phi_1\|_\infty}{\|\phi_2\|_\infty}} \frac{1}{\frac{\|\phi_2\|_\infty}{\|\phi_3\|_\infty}} \cdots \frac{1}{\frac{\|\phi_{n-1}\|_\infty}{\|\phi_n\|_\infty}} = \|\phi_n\|_\infty.$$

Thus, it satisfies the nonlinear problem

$$\begin{cases} -\Delta_p u_{n+1} = \gamma_n u_n^{p-1} & \text{in } B, \\ u_n = 0 & \text{on } \partial B, \end{cases} \tag{23}$$

and is decreasing. Therefore, arguing as in the proof of Proposition 3.2, we can pass the limit in (23) in order to obtain (22). However, differently from the sequence of Proposition 3.2, we have in addition

$$\|u_n\|_\infty = 1$$

for every $n \in \mathbb{N}$, which allows us to conclude that $\|u\|_\infty = \lim \|u_n\|_\infty = 1$, hence u is not the null function and thus $\gamma = \lambda_p$ (see remark after Proposition 3.2). $\square$

Next we will show that the sequence (ϕ_n/ϕ_{n+1}) converges uniformly to the constant function λ_p on each compact set contained in B.

Lemma 4.5. *For each $0 < \varepsilon < 1$ define*

$$K_\varepsilon := \left(\int_{1-\varepsilon}^{1} \left(\frac{\varepsilon}{\theta}\right)^{\frac{N-1}{p-1}} d\theta\right)^{-1}.$$

Then

$$0 \leqslant \left(\frac{\phi_n}{\phi_{n+1}}\right)' \leqslant K_\varepsilon \left\|\frac{\phi_1}{\phi_2}\right\|_\infty \quad \textit{on the interval } [\varepsilon, 1-\varepsilon]. \tag{24}$$

Proof. Since (see Propositions 2.6 and 2.7)

$$\left\|\frac{\phi_n}{\phi_{n+1}}\right\|_\infty \leqslant \left\|\frac{\phi_1}{\phi_2}\right\|_\infty, \quad \text{if } n \geqslant 2 \tag{25}$$

and

$$0 \leqslant \left(\frac{\phi_n}{\phi_{n+1}}\right)' = \frac{\phi_{n+1}\phi_n' - \phi_n\phi_{n+1}'}{\phi_{n+1}^2} = \frac{\phi_n}{\phi_{n+1}}\frac{|\phi_{n+1}'|}{\phi_{n+1}} - \frac{|\phi_n'|}{\phi_n} \leqslant \frac{\phi_n}{\phi_{n+1}}\frac{|\phi_{n+1}'|}{\phi_{n+1}},$$

it suffices to show that

$$\frac{|\phi_{n+1}'|}{\phi_{n+1}} \leqslant K_\varepsilon \quad \text{in } [\varepsilon, 1-\varepsilon].$$

And indeed, for $\varepsilon \leqslant r \leqslant 1-\varepsilon < 1$ we have

$$\begin{aligned}
\phi_{n+1}(r) &= \int_r^1 \left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \phi_n^{p-1}(s)\,ds\right)^{\frac{1}{p-1}} d\theta \\
&\geqslant \left(\int_r^1 \theta^{-\frac{N-1}{p-1}}\,d\theta\right)\left(\int_0^r s^{N-1}\phi_n^{p-1}(s)\,ds\right)^{\frac{1}{p-1}} \\
&= \left(\int_{1-\varepsilon}^1 \theta^{-\frac{N-1}{p-1}}\,d\theta\right) r^{\frac{N-1}{p-1}} \left|\phi_{n+1}'(r)\right| \\
&\geqslant \left(\int_{1-\varepsilon}^1 \left(\frac{\varepsilon}{\theta}\right)^{\frac{N-1}{p-1}} d\theta\right) \left|\phi_{n+1}'(r)\right|. \qquad \square
\end{aligned}$$

Theorem 4.6. *For each fixed* $0 < \varepsilon < 1$ *we have*

$$\left[\frac{\phi_n(|x|)}{\phi_{n+1}(|x|)}\right]^{\frac{1}{p-1}} \to \lambda_p$$

uniformly on the annulus $\Omega_\varepsilon^{1-\varepsilon} := \{\varepsilon < |x| < 1-\varepsilon\} \subset B_1$.

Proof. In view of (24) and (25), it follows from Arzela–Ascoli's theorem that, up to a subsequence, $(\frac{\phi_n(r)}{\phi_{n+1}(r)})$ converges uniformly to a function $w \in C([\varepsilon, 1-\varepsilon])$.

Taking $u_n(|x|) = \frac{\phi_n(|x|)}{\|\phi_n\|_\infty}$ as in the proof of Theorem 4.4, we can write

$$-\Delta_p u_{n+1} = \frac{\phi_n^{p-1}}{\|\phi_{n+1}\|_\infty^{p-1}} = \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} u_{n+1}^{p-1} \quad \text{in } \Omega_\varepsilon^{1-\varepsilon}.$$

Thus, passing the limit in this equation for a convenient subsequence, we obtain

$$-\Delta_p u = w^{p-1} u,$$

where u is the eigenfunction given in Theorem 4.4. Therefore,

$$w^{p-1} \equiv \lambda_p,$$

for $-\Delta_p u = \lambda_p u^{p-1}$.

The proof is complete since the limit is independent of the particular subsequence that converges to w. □

Corollary 4.7.

$$\Gamma = \lambda_p.$$

Proof. We have

$$\Gamma_n = \sup_{0 \leqslant r \leqslant 1} \left(\frac{\phi_n}{\phi_{n+1}}\right)^{p-1} = \lim_{r \to 1^+} \left(\frac{\phi_n(r)}{\phi_{n+1}(r)}\right)^{p-1} = \left(\frac{\phi_n'(1)}{\phi_{n+1}'(1)}\right)^{p-1}$$

and

$$\begin{aligned}\left(\frac{\phi_n'(1)}{\phi_{n+1}'(1)}\right)^{p-1} &= \frac{\int_0^1 s^{N-1} \phi_{n-1}^{p-1}(s)\, ds}{\int_0^1 s^{N-1} \phi_n^{p-1}(s)\, ds} \\ &= \frac{\int_0^1 s^{N-1} (\frac{\phi_{n-1}}{\phi_n}(s))^{p-1} (\frac{\phi_n(s)}{\|\phi_n\|_\infty})^{p-1}\, ds}{\int_0^1 s^{N-1} (\frac{\phi_n(s)}{\|\phi_n\|_\infty})^{p-1}\, ds}.\end{aligned}$$

Since $\frac{\phi_{n-1}}{\phi_n}$ and $\frac{\phi_n}{\|\phi_n\|_\infty}$ are bounded, $\lambda_p = \lim(\frac{\phi_{n-1}}{\phi_n})^{p-1}$ and $u(|x|) = \lim(\frac{\phi_n(|x|)}{\|\phi_n\|_\infty})^{p-1}$ is the eigenfunction obtained above, we can apply Lebesgue's dominated convergence theorem to obtain

$$\Gamma = \lim\left(\frac{\phi_n'(1)}{\phi_{n+1}'(1)}\right)^{p-1} = \frac{\int_0^1 s^{N-1} \lambda_p^{p-1} u^{p-1}(s)\, ds}{\int_0^1 s^{N-1} u^{p-1}(s)\, ds} = \lambda_p. \qquad \square$$

Corollary 4.8.

$$\nu := \lim\left(\frac{\|\phi_n\|_p}{\|\phi_{n+1}\|_p}\right)^{p-1} = \lambda_p.$$

Proof. In view of the last corollary, this follows from Corollary 2.9. □

This result in fact holds for any L^q-norm ($q > 1$):

Corollary 4.9. *For any $q > 1$ there holds*

$$\lim\left(\frac{\|\phi_n\|_q}{\|\phi_{n+1}\|_q}\right)^{p-1} = \lambda_p.$$

Proof. As before set $u(|x|) = \lim(\frac{\phi_n(|x|)}{\|\phi_n\|_\infty})^{p-1}$. Then it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\lim\left(\frac{\|\phi_n\|_q}{\|\phi_{n+1}\|_q}\right)^{p-1} &= \lim\left(\frac{\int_0^1 s^{N-1}\phi_n^q(s)\,ds}{\int_0^1 s^{N-1}\phi_{n+1}^q(s)\,ds}\right)^{\frac{p-1}{q}} \\
&= \lim\left(\frac{\|\phi_n\|_\infty}{\|\phi_{n+1}\|_\infty}\right)^{p-1}\left(\frac{\int_0^1 s^{N-1}\lim(\frac{\phi_n(s)}{\|\phi_n\|_\infty})^q\,ds}{\int_0^1 s^{N-1}\lim(\frac{\phi_{n+1}(s)}{\|\phi_{n+1}\|_\infty})^q\,ds}\right)^{\frac{p-1}{q}} \\
&= \lambda_p\left(\frac{\int_0^1 s^{N-1}u^q(s)\,ds}{\int_0^1 s^{N-1}u^q(s)\,ds}\right)^{\frac{p-1}{q}} \\
&= \lambda_p. \qquad \Box
\end{aligned}$$

As at the end of last section, we also remark that all results in this section remain valid if we consider a radially symmetric weight $\omega(|x|)$. Moreover, the results can be naturally extended to the Dirichlet problem in $\mathbb{R}^N$ with an appropriate weight.

5. The case $p = 2$

In this section, we give a complete proof of the convergence of the three sequences to the first eigenvalue of the Laplacian.

Let

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$$

be the increasing sequence of Dirichlet eigenvalues for the Laplacian $-\Delta$ in Ω and $(e_n)_{n\in\mathbb{N}} \subset W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ be a corresponding sequence of eigenfunctions which is also an orthogonal system for $L^2(\Omega)$ and normalized by the sup-norm, that is, $\|e_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Denote the inner product in $L^2(\Omega)$ by

$$\langle u, v\rangle = \int_\Omega uv\,dx.$$

If $\xi \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ is such that $\xi > 0$ in Ω, then

$$\xi = \sum_{k=1}^{\infty} \alpha_k e_k \tag{26}$$

with

$$\alpha_k = \langle \xi, e_k \rangle, \quad k = 1, 2, \ldots,$$

and we may assume

$$\alpha_1 = \langle \xi, e_1 \rangle = \int_{\Omega} \xi e_1 \, dx > 0,$$

since we can take $e_1 > 0$ in Ω. Moreover,

$$\sum_{k=2}^{\infty} \alpha_k^2 \|e_k\|^2 = \|\xi\|_2^2 - \alpha_1^2 \|e_1\|^2 < \|\xi\|_2^2.$$

Now, if $\phi \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ is such that

$$\begin{cases} -\Delta\phi = \xi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

it follows that

$$\phi = \sum_{k=1}^{\infty} \langle \phi, e_k \rangle e_k$$

with

$$\begin{aligned} \sum_{k=1}^{\infty} \alpha_k e_k = \xi = -\Delta\phi &= \sum_{k=1}^{\infty} \langle \phi, e_k \rangle (-\Delta e_k) \\ &= \sum_{k=1}^{\infty} \lambda_k \langle \phi, e_k \rangle e_k, \end{aligned}$$

whence

$$\langle \phi, e_k \rangle = \frac{\alpha_k}{\lambda_k}.$$

Thus,

$$\phi = \sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k} e_k.$$

Returning to sequence (ϕ_n), if

$$1 = \sum_{k=1}^{\infty} \alpha_k e_k$$

is the expansion of the function $\xi \equiv 1$, we obtain recursively

$$\phi_n = \sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k^n} e_k,$$

that is,

$$\phi_n = \frac{1}{\lambda_1^n}(\alpha_1 e_1 + \psi_n),$$

where

$$\psi_n := \sum_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_k}\right)^n \alpha_k e_k.$$

Theorem 5.1. $\lambda_1 = \lim \frac{\|\phi_n\|_2}{\|\phi_{n+1}\|_2}$.

Proof. We assert that if $(\psi_n)_{n\in\mathbb{N}}$ is the sequence defined above, then

$$\psi_n \to 0 \quad \text{in } L^2(\Omega).$$

Indeed, this follows immediately from the estimate

$$\|\psi_n\|_2^2 = \sum_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_k}\right)^{2n} \alpha_k^2 \|e_k\|_2^2 \leqslant \left(\frac{\lambda_1}{\lambda_2}\right)^{2n} \sum_{k=2}^{\infty} \alpha_k^2 \|e_k\|_2^2 \leqslant \|1\|_2 \left(\frac{\lambda_1}{\lambda_2}\right)^n$$

and the fact that $\lambda_1 < \lambda_2$.

Therefore,

$$\lim \frac{\|\phi_n\|_2}{\|\phi_{n+1}\|_2} = \lim\left(\lambda_1 \frac{\|\alpha_1 e_1 + \psi_n\|_2}{\|\alpha_1 e_1 + \psi_{n+1}\|_2}\right) = \lambda_1 \lim\left(\frac{\alpha_1^2 \|e_1\|_2^2 + \|\psi_n\|_2^2}{\alpha_1^2 \|e_1\|_2^2 + \|\psi_{n+1}\|_2^2}\right)^{\frac{1}{2}} = \lambda_1. \qquad \square$$

Theorem 5.2. $\psi_n \to 0$ *uniformly in* Ω.

Proof. Since the convergence of the eigenfunction expansion of $\xi \equiv 1$ is absolute, let

$$M := \sum_{k=1}^{\infty} |\alpha_k|.$$

Then

$$\|\psi_n\|_\infty = \left\| \sum_{k=2}^{\infty} \left(\frac{\lambda_1}{\lambda_k}\right)^n \alpha_k e_k \right\|_\infty \leqslant M \left(\frac{\lambda_1}{\lambda_2}\right)^n$$

and the result follows letting $n \to \infty$. □

Corollary 5.3. $\frac{\phi_n}{\phi_{n+1}} \to \lambda_1$ *uniformly in any compact subset* $K \Subset \Omega$.

Proof. Let $K \Subset \Omega$ be a compact set and set

$$m_K := \min_K e_1 > 0.$$

Since $\psi_n \to 0$ uniformly in Ω, we have for all sufficiently large n that

$$\begin{aligned} \alpha_1 m_K \leqslant \alpha_1 e_1 &\leqslant |\alpha_1 e_1 + \psi_{n+1}| + |\psi_{n+1}| \\ &< |\alpha_1 e_1 + \psi_{n+1}| + \frac{\alpha_1}{2} m_K, \end{aligned}$$

whence

$$|\alpha_1 e_1 + \psi_{n+1}| \geqslant \frac{\alpha_1}{2} m_K.$$

Thus, on K we obtain

$$\begin{aligned} \left| \frac{\phi_n}{\phi_{n+1}} - \lambda_1 \right| &= \lambda_1 \left| \frac{\alpha_1 e_1 + \psi_n}{\alpha_1 e_1 + \psi_{n+1}} - 1 \right| \\ &= \lambda_1 \left| \frac{\psi_n - \psi_{n+1}}{\alpha_1 e_1 + \psi_{n+1}} \right| \\ &\leqslant \frac{2\lambda_1}{\alpha_1 m_K} |\psi_n - \psi_{n+1}| \\ &\to 0 \end{aligned}$$

uniformly. □

6. Higher eigenvalues

In the case $p = 2$, higher eigenvalues and their respective eigenfunctions can also in principle be obtained by this technique. Suppose now that the first nonzero coefficient of $\xi \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ is α_{k_0} for some $k_0 > 1$, that is,

$$\xi = \sum_{k=k_0}^{\infty} \alpha_k e_k \tag{27}$$

with

$$\alpha_k = \langle v, e_k \rangle, \quad k = k_0, k_0 + 1, \ldots.$$

Then

$$\phi_n = \sum_{k=k_0}^{\infty} \frac{\alpha_k}{\lambda_k^n} e_k,$$

which we write in the form

$$\phi_n = \frac{1}{\lambda_{k_0}^n} (\alpha_{k_0} e_{k_0} + \psi_n), \tag{28}$$

where now

$$\psi_n := \sum_{k=k_0+1}^{\infty} \left(\frac{\lambda_{k_0}}{\lambda_k} \right)^n \alpha_k e_k.$$

Following the same steps of the previous section, we can conclude that $\psi_n \to 0$ in $L^2(\Omega)$ and that

$$\lambda_{k_0} = \lim \frac{\|\phi_n\|_2}{\|\phi_{n+1}\|_2}.$$

Moreover, choosing ξ sufficiently regular so that the series of the coefficients

$$M := \sum_{k=1}^{\infty} |\alpha_k|$$

is absolutely convergent, we also have

$$\|\psi_n\|_\infty = \left\| \sum_{k=k_0+1}^{\infty} \left(\frac{\lambda_{k_0}}{\lambda_k} \right)^n \alpha_k e_k \right\|_\infty \leqslant M \left(\frac{\lambda_{k_0}}{\lambda_{k_0+1}} \right)^n,$$

that is, $\psi_n \to 0$ uniformly in Ω, which implies as above that $\frac{\phi_n}{\phi_{n+1}}$ converges uniformly to the constant function λ_{k_0} in compact subsets of $K \Subset \Omega \cap \operatorname{supp}(e_{k_0})$.

7. Numerical results

In this section we present some of the numerical results which we were able to compute for some domains. We compare them with results obtained elsewhere. Computations were performed on a Windows XP/Pentium 4 – 2.8 GHz platform, using the GCC compiler.

Table 1
First eigenvalue for the p-Laplacian on the unit ball.

p	$N=2$	$N=3$	$N=4$	p	$N=2$	$N=3$	$N=4$
1.1	2.5694	3.8728	5.1871	2.6	8.1192	15.0590	24.0121
1.2	2.9656	4.5151	6.1020	2.7	8.5355	16.0412	25.8617
1.3	3.3263	5.1283	7.0064	2.8	8.9598	17.0586	27.8027
1.4	3.6741	5.7431	7.9390	2.9	9.3921	18.1117	29.8374
1.5	4.0180	6.3717	8.9154	3.0	9.8324	19.2013	31.9687
1.6	4.3624	7.0201	9.9443	3.1	10.2809	20.3278	34.1991
1.7	4.7098	7.6920	11.0314	3.2	10.7375	21.4917	36.5314
1.8	5.0619	8.3898	12.1810	3.3	11.2022	22.6937	38.9681
1.9	5.4195	9.1153	13.3969	3.4	11.6751	23.9341	41.5120
2.0	5.7835	9.8698	14.6822	3.5	12.1561	25.2136	44.1659
2.1	6.1543	10.6545	16.0400	3.6	12.6453	26.5327	46.9325
2.2	6.5321	11.4701	17.4730	3.7	13.1427	27.8919	49.8144
2.3	6.9174	12.3177	18.9841	3.8	13.6482	29.2916	52.8146
2.4	7.3103	13.1979	20.5759	3.9	14.1619	30.7325	55.9359
2.5	7.7108	14.1115	22.2510	4.0	14.6838	32.2150	59.1810

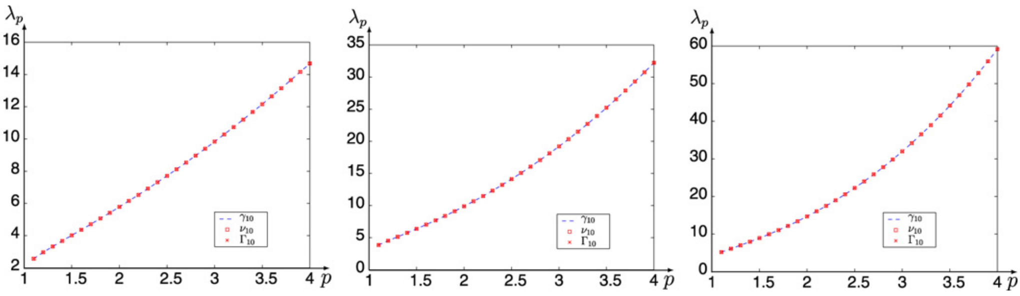

Fig. 1. Graphs of p ($1 < p \leqslant 4$) versus values of γ_{10}, ν_{10} and Γ_{10} for the N-dimensional unit ball and $N=2$ (left), $N=3$ (center), $N=4$ (right).

7.1. The unit ball

In order to compute the value of the first eigenvalue for the p-Laplacian in the unit ball, we mixed the composite Simpson and trapezoidal methods for computation of the associated integrals in the expression of ν_n. In Table 1, the results for the first eigenvalue of the p-Laplacian for values of p ranging from 1.1 to 4.0 for balls of dimensions $N = 2, 3, 4$, are displayed, truncated at the fourth decimal place, after 10 iterations. The results are also visually displayed in Fig. 1. For comparison, the known value of the first eigenvalue for the Laplacian on the unit bidimensional ball is 5.7832, which means that our error should be about 0.04%. This result compares well with the one obtained in [8], where a 1.3% precision was attained.

7.2. The unit square

In order to solve the p-Laplacian in the unit square $[0, 1] \times [0, 1]$ we used the algorithm proposed in [3], coupled with the homotopy perturbation method (HPM) of [6] for the exact line searches in the nonlinear conjugate gradient method. In Table 2 we see the values for the first eigenvalue of the p-Laplacian for values of p ranging from $p = 2$ to $p = 3$,

Table 2
First eigenvalue for the p-Laplacian on the unit square.

p	γ_5	ν_5	Γ_5
2.0	19.7145	19.7348	19.9270
2.1	22.3239	22.3460	22.4447
2.2	25.2168	25.2412	25.3343
2.3	28.2413	28.4495	28.6139
2.4	31.9750	32.0024	32.5685
2.5	35.5746	35.9344	37.6961
2.6	38.5547	40.2827	40.8167
2.7	41.4917	45.0890	52.8657
2.8	5.5593	50.3972	642.6432
2.9	7.8823	56.2567	670.7254
3.0	14.6719	62.7208	205.0535

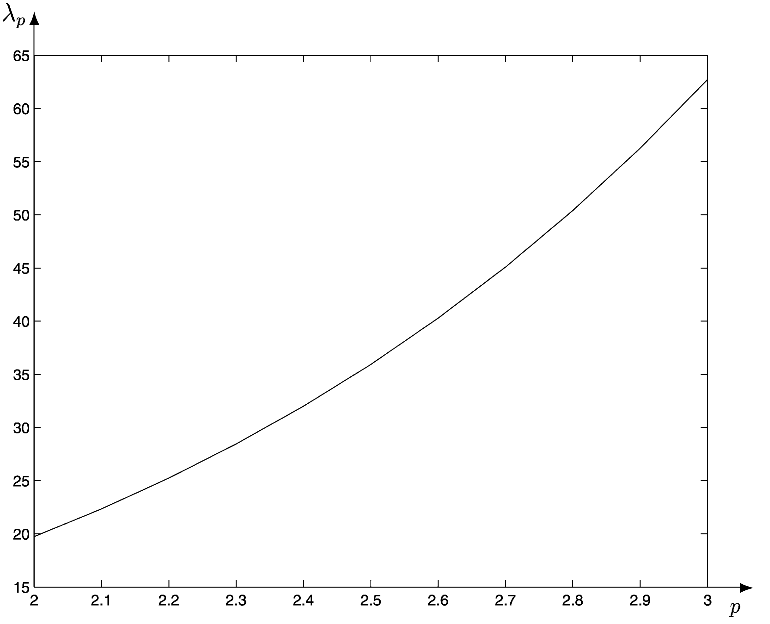

Fig. 2. Graph of p ($2 \leqslant p \leqslant 3$) versus ν_5 for the square $[0, 1] \times [0, 1]$.

with 0.1 increment, truncated at the fourth decimal place, after 5 iterations, for all three sequences. We see that sequence (ν_n) has a much faster rate of convergence and less numerical error, especially as the value of p increases. For comparison, the known value of the first eigenvalue for the Laplacian on the unit square is $2\pi^2 = 19.7392$, which means that the error in ν_5 is about 0.02%. Fig. 2 displays the results for ν_5. This result with only 5 iterations compares very favourably with the one obtained in [8], where a 3% precision was attained.

It must be remarked that better and faster results should be obtainable with more precise and faster methods for solving the p-Laplacian equation in each iteration.

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