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MATHEMATICAL PROBLEMS OF NONLINEARITY

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A Note on Tonelli Lagrangian Systems on $\mathbb{T}^2$ with Positive Topological Entropy on a High Energy Level

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In this work we study the dynamical behavior of Tonelli Lagrangian systems defined on the tangent bundle of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We prove that the Lagrangian flow restricted to a high energy level $E_L^{-1}(c)$ (i.e., $c > c_0(L)$) has positive topological entropy if the flow satisfies the Kupka-Smale property in $E_L^{-1}(c)$ (i.e., all closed orbits with energy c are hyperbolic or elliptic and all heteroclinic intersections are transverse on $E_L^{-1}(c)$). The proof requires the use of well-known results from Aubry – Mather theory.

Keywords: Tonelli Lagrangian system, Aubry – Mather theory, static classes

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1. Introduction

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be endowed with a Riemannian metric $\langle\cdot,\cdot\rangle$. A Tonelli Lagrangian on $\mathbb{T}^2$ is a smooth function $L\colon T\mathbb{T}^2 \to \mathbb{R}$ that satisfies two conditions:

- *convexity*: for each fiber $T_x\mathbb{T}^2 \cong \mathbb{R}^2$, the restriction $L(x,\cdot)\colon T_x\mathbb{T}^2 \to \mathbb{R}$ has a positive defined Hessian,
- *superlinearity*: $\lim_{\|v\|\to\infty} \dfrac{L(x,v)}{\|v\|} = \infty$, uniformly in $x \in \mathbb{T}^2$.

The action of L over an absolutely continuous curve $\gamma\colon [a,b] \to \mathbb{T}^2$ is defined by

$$A_L(\gamma) = \int\limits_a^b L(\gamma(t),\dot\gamma(t))\,dt.$$

The extremal curves for the action are given by solutions of the *Euler – Lagrange equation* which in local coordinates can be written as

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v} = 0.$$

The *Lagrangian flow* $\phi_t\colon T\mathbb{T}^2 \to T\mathbb{T}^2$ is defined by $\phi_t(x,v) = (\gamma(t),\dot\gamma(t))$ where $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ is the solution of the *Euler – Lagrange equation*, with the initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = v$.

The *energy function* $E_L\colon T\mathbb{T}^2 \to \mathbb{R}$ is defined by

$$E_L(x,v) = \left\langle \frac{\partial L}{\partial v}(x,v), v \right\rangle - L(x,v). \tag{1.1}$$

The subsets $E_L^{-1}(c) \subset T\mathbb{T}^2$ are called *energy levels* and they are invariant under the Lagrangian flow of L. Note that the superlinearity condition implies that any nonempty energy levels are compact. Therefore, the flow ϕ_t is defined for all $t \in \mathbb{R}$.

The Lagrangian flow of L is conjugated to a Hamiltonian flow on $T^*\mathbb{T}^2$, with the canonical symplectic structure, by the Legendre transformation $\mathcal{L}\colon T\mathbb{T}^2 \to T^*\mathbb{T}^2$ given by

$$\mathcal{L}(x,v) = \left(x, \frac{\partial L}{\partial v}(x,v)\right).$$

The corresponding Hamiltonian $H\colon T^*\mathbb{T}^2 \to \mathbb{R}$ is

$$H(x,p) = \max_{v\in T_x\mathbb{T}^2}\{p(v) - L(x,v)\}$$

and we have the Fenchel inequality

$$p(v) \leqslant H(x,p) + L(x,v)$$

with equality if only if $(x,p) = \mathcal{L}(x,v)$, or equivalently, $p = \dfrac{\partial L}{\partial v}(x,v) \in T_x^*\mathbb{T}^2$. Therefore, by (1.1),

$$H\left(x, \frac{\partial L}{\partial v}(x,v)\right) = E(x,v).$$

Given a nonempty energy level $E_L^{-1}(c)$, the set $H^{-1}(c) := \mathcal{L}\left(E_L^{-1}(c)\right) \subset T^*\mathbb{T}^2$ is called the *Hamiltonian level*.

We denote by $h_{top}(L,c)$ the topological entropy of the Lagrangian flow $\phi_t|_{E_L^{-1}(c)}$, for any nonempty energy level $E_L^{-1}(c)$. The *topological entropy* is an invariant that, roughly speaking, measures the complexity of its orbits structure. The relevant question about the topological entropy is whether it is positive or vanishes. Namely, if $\theta \in E_L^{-1}(c)$ and $T, \delta > 0$, we define the (δ, T)-*dynamical ball* centered at θ as

$$B(\theta,\delta,T) = \{v \in E_L^{-1}(c)\colon d(\phi_t(v),\phi_t(\theta)) < \delta \text{ for all } t \in [0,T]\},$$

where d is the distance function in $E_L^{-1}(c)$. Let $N_\delta(T)$ be the minimal quantity of the (δ, T)-dynamical ball needed to cover $E_L^{-1}(c)$. The topological entropy is the limit $\delta \to 0$ of the exponential growth rate of $N_\delta(T)$, that is,

$$h_{top}(L,c) := \lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T}\log N_\delta(T).$$

Thus, if $h_{top}(L,c) > 0$, some dynamical balls must contract exponentially at least in one direction.

For example, if $\langle\cdot,\cdot\rangle$ denotes the flat metric and $L(x,v) = 1/2\langle v,v\rangle$, then the corresponding Lagrangian flow is the geodesic flow on the flat torus, which is given by

$$\phi_t(x,v) = (x + tv \mod \mathbb{Z}^2, v).$$

It follows from straight computations that $h_{top}(L,c) = 0$ for all $c > 0$. In this example, the corresponding Hamiltonian flow is integrable. For an integrable Hamiltonian system on a four-dimensional symplectic manifold under certain regularity assumptions (see [13]), the topological entropy of the Hamiltonian flow restricted to a regular compact energy level vanishes.

In [16], J. P. Schröder obtained a partial answer about the integrability reverse claim of Paternain's theorem[13], which is false in general, as has been known for a long time [6]. Schröder proved that, if the topological entropy of the Lagrangian flow on the level above the Mañé critical value vanishes, then, for all directions $\zeta \in \mathbb{S}^1$, there are invariant Lipschitz graphs $\mathcal{T}_\zeta$ (ζ with irrational slope), $\mathcal{T}_\zeta^\pm$ (ζ with rational slope) over $\mathbb{T}^2$, contained in $\{E = e\}$ whose complement of its union is a tubular neighborhood of $\mathcal{T}_\zeta$ and the lifted orbits from $\mathcal{T}_\zeta$, $\mathcal{T}_\zeta^\pm$ on the universal cover $\mathbb{R}^2$ are going to ∞, i.e., heteroclinic orbits. He used the gap condition to prove that, if in a strip between two neighboring periodic minimizers no foliation by heteroclinic minimizers exists, then there are instability regions which imply in its turn positive entropy.

The Mañé *strict critical value* of L is the real number $c_0(L)$ given by

$$c_0(L) = \inf\{k \in \mathbb{R}\colon A_{L+k}(\gamma) \geqslant 0, \text{ for all contractible closed curves on } \mathbb{T}^2\}.^1 \tag{1.2}$$

Theorem 1. *Let $L\colon T\mathbb{T}^2 \to \mathbb{R}$ be a Tonelli Lagrangian and let $c > c_0(L)$. Suppose that the Lagrangian flow restricted to an energy level $E_L^{-1}(c)$ satisfies the following conditions:*

1) *all closed orbits in $E_L^{-1}(c)$ are hyperbolic or elliptic, and*

2) *all heteroclinic intersections in $E_L^{-1}(c)$ are transverse.*

Then $h_{top}(L,c) > 0$.

[1] On an arbitrary closed manifold, this equality defines the universal critical value $c_u(L)$. The value $c_0(L)$ is the infimum value of $k \in \mathbb{R}$ such that the $L + k$ action is positive on the set of all closed curves that are homologous to zero. Of course, $c_u(L) \leqslant c_0(L)$, and $c_u(L) = c_0(L)$ for systems on $\mathbb{T}^2$.

Let $C^r(\mathbb{T}^2)$ be the set of potentials $u\colon \mathbb{T}^2 \to \mathbb{R}$ of class C^r endowed with the C^r-topology. We recall that a subset $\mathcal{O} \subset \mathcal{C}^r(\mathbb{T}^2)$ is called *residual* if it contains a countable intersection of open and dense subsets. In [12], E. Oliveira proved a version of the Kupka – Smale theorem for the Tonelli Hamiltonian and Lagrangian systems on any closed surfaces. More precisely, it follows from [12] that, for each $c \in \mathbb{R}$, there exists a residual set $\mathcal{KS}(c) \subset \mathcal{C}^r(\mathbb{T}^2)$ such that every Hamiltonian $H_u = H + u$, with $u \in \mathcal{KS}$, satisfies the Kupka – Smale property, i.e., all closed orbits with energy c are hyperbolic or elliptic and all heteroclinic intersections are transverse on $E_L^{-1}(c)$. See also [15], where L. Rifford and R. Ruggiero proved the Kupka – Smale theorem for Tonelli Lagrangian systems on closed manifolds of any dimension.

So, if we take the residual subset $\mathcal{KS}(c) \subset \mathcal{C}^r(\mathbb{T}^2)$ given by the Kupka – Smale theorem, and by the continuity of the critical values (cf. Lemma 5.1 in [5]) , we have the following corollary.

Corollary 1. *Given $c > c_0(L)$, there exists a smooth potential $u\colon \mathbb{T}^2 \to \mathbb{R}$ of C^r-norm arbitrarily small (for any $r \geqslant 2$) such that $h_{top}(L - u, c) > 0$.*

2. The Mather and Aubry sets

In this section we recall the definitions of the Mather sets and Aubry sets for the case of an (autonomous) Tonelli Lagrangian on the torus. Aubry – Mather theory was introduced by J. Mather in [10, 11] for convex, superlinear and time-periodic Lagrangian systems on any closed manifolds. Details and proofs of the main results of this section can be seen in the original works of J. Mather cited above.

Let us recall the main concepts introduced by J. Mather in [10]. We denote by $\mathfrak{B}(L)$ the set of all Borel probability measures, with compact support, that are invariant under the Lagrangian flow of L. By duality, given $\mu \in \mathfrak{B}(L)$, there is a unique homology class $\rho(\mu) \in H_1(\mathbb{T}^2, \mathbb{R})$ such that

$$\langle \rho(\mu), [\omega] \rangle = \int\limits_{T\mathbb{T}^2} \omega \, d\mu, \tag{2.1}$$

for any closed 1-form ω on $\mathbb{T}^2$.

Then the Mather *β-function* is defined by

$$\beta(h) = \inf \left\{ \int\limits_{T\mathbb{T}^2} L \, d\mu \colon \mu \in \mathfrak{B}(L) \text{ with } \rho(\mu) = h \right\}.$$

The function $\beta\colon H_1(M, \mathbb{R}) \to \mathbb{R}$ is convex and superlinear. A measure $\mu \in \mathfrak{B}(L)$ that satisfies

$$\int\limits_{T\mathbb{T}^2} L \, d\mu = \beta(\rho(\mu))$$

is called a *$\rho(\mu)$-minimizing measure*.

The Mather *α-function* can be defined as the convex dual (or conjugate) function of β, i.e., $\alpha = \beta^*\colon H^1(M, \mathbb{R}) \to \mathbb{R}$ is given by the so-called Fenchel transformation

$$\alpha([\omega]) = \sup_{h \in H_1(M,\mathbb{R})} \{\langle [\omega], h \rangle - \beta(h)\} = - \inf_{\mu \in \mathfrak{B}(L)} \left\{ \int\limits_{TM} L - \omega \, d\mu \right\}.$$

By convex duality, we have that α is also convex and superlinear, and $\alpha^* = \beta$. Moreover, a measure μ_0 is $\rho(\mu_0)$-minimizing if and only if there is a closed 1-form ω_0, such that

$$\int_{TM} L - \omega_0 \; d\mu_0 = -\alpha([\omega_0]).$$

Such a class $[\omega_0] \in H^1(M, \mathbb{R})$ is called a *subderivative* of β at the point $\rho(\mu_0)$.

We say that $\mu \in \mathfrak{B}(L)$ is a $[\omega]$-*minimizing measure* of L if

$$\int_{T\mathbb{T}^2} L - \omega \; d\mu = \min \left\{ \int_{T\mathbb{T}^2} L - \omega \; d\nu \colon \nu \in \mathfrak{B}(L) \right\} = -\alpha([\omega]).$$

Let $\mathfrak{M}_L([\omega]) \subset \mathfrak{B}(L)$ be the set of all $[\omega]$-minimizing measures (it only depends on the cohomology class $[\omega]$). The ergodic components of a $[\omega]$-minimizing measure are also $[\omega]$-minimizing measures, so the set $\mathfrak{M}_L([\omega])$ is a simplex whose extremal measures are ergodic $[\omega]$-minimizing measures. In particular $\mathfrak{M}_L([\omega])$ is a compact subset of $\mathfrak{B}(L)$ with the weak*-topology.

For each $[\omega] \in H^1(\mathbb{T}^2, \mathbb{R})$, we define the *Mather set of cohomology class* $[\omega]$ as

$$\widetilde{\mathcal{M}}_L([\omega]) = \overline{\bigcup_{\mu \in \mathfrak{M}_L([\omega])} Supp(\mu)}.$$

We set $\pi(\widetilde{\mathcal{M}}_L([\omega])) = \mathcal{M}_L([\omega])$, and call it the *projected Mather set*, where $\pi \colon T\mathbb{T}^2 \to \mathbb{T}^2$ denotes the canonical projection. The celebrated *graph theorem* proved by J. Mather in [10, Theorem 2], asserts that $\widetilde{\mathcal{M}}_L([\omega])$ is nonempty, compact, invariant under the Euler – Lagrange flow and the map $\pi|_{\widetilde{\mathcal{M}}_L([\omega])} \colon \widetilde{\mathcal{M}}_L([\omega]) \to \mathcal{M}_L([\omega])$ is a bi-Lipschitz homeomorphism.

Following J. Mather in [11], for $t > 0$ and $x, y \in \mathbb{T}^2$, define the *action potential* for the Lagrangian deformed by a closed 1-form ω as

$$\Phi_\omega(x, y, t) = \inf \left\{ \int_0^t L(\gamma(s), \dot\gamma(s)) - \omega_{\gamma(s)}(\dot\gamma(s)) \, ds \right\},$$

where the infimum is taken over all absolutely continuous curves $\gamma \colon [0, t] \to \mathbb{T}^2$ such that $\gamma(0) = x$ and $\gamma(t) = y$. The infimum is in fact a minimum by Tonelli's theorem.

We define the *Peierls barrier* for the Lagrangian $L - \omega$ as the function $h_\omega \colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ given by

$$h_\omega(x, y) = \liminf_{t \to +\infty} \left\{ \Phi_\omega(x, y, t) + \alpha([\omega])t \right\},$$

and the *projected Aubry set for the cohomology class* $[\omega] \in H^1(\mathbb{T}^2, \mathbb{R})$ as

$$\mathcal{A}_L([\omega]) = \big\{ x \in \mathbb{T}^2 \colon \; h_\omega(x, x) = 0 \big\}.$$

By symmetrizing h_ω, we define the *semidistance* $\delta_{[\omega]}$ on $\mathcal{A}_L([\omega])$:

$$\delta_{[\omega]}(x, y) = h_\omega(x, y) + h_\omega(y, x).$$

This function $\delta_{[\omega]}$ is nonnegative and satisfies the triangle inequality.

We define the *Aubry set* (that is also called static set) as the invariant set

$$\widetilde{\mathcal{A}}_L([\omega]) = \big\{(x,v) \in T\mathbb{T}^2 \colon \pi \circ \phi_t(x,v) \in \mathcal{A}_L([\omega]),\ \forall\ t \in \mathbb{R}\big\}.$$

By definition, $\pi(\widetilde{\mathcal{A}}_L([\omega])) = \mathcal{A}_L([\omega])$. In [11, Theorem 6.1], J. Mather proved that this set is compact, $\widetilde{\mathcal{M}}_L([\omega]) \subseteq \widetilde{\mathcal{A}}_L([\omega])$ and the extension of the graph theorem to the Aubry set, i.e., the mappings $\pi|_{\widetilde{\mathcal{A}}_L([\omega])} \colon \widetilde{\mathcal{A}}_L([\omega]) \to \mathcal{A}_L([\omega])$ is a bi-Lipschitz homeomorphism.

Finally, we define the *Mañé set of cohomology class* $[\omega]$, which we denote by $\widetilde{\mathcal{N}}_L([\omega])$, as the set of orbits $\phi_t(\theta) = (\gamma(t), \dot\gamma(t)) \in T\mathbb{T}^2$ such that, for all $a < b \in \mathbb{R}$, the trajectories $\gamma \colon \mathbb{R} \to \mathbb{T}^2$ satisfy

$$\int\limits_a^b L(\gamma(s), \dot\gamma(s)) - \omega_{\gamma(s)}(\dot\gamma(s)) + \alpha([\omega])\ ds = \inf_{t>0} \{\Phi_\omega(\gamma(a), \gamma(b), t) + \alpha([\omega])t\}.$$

These curves, which satisfy the above equality, are also called semistatic curves or $[\omega]$-minimizing curves.

Let us now state some important properties and results on these invariant sets that we going to use in the proof of Theorem 1.

Using the Mather α-function, we have the following equivalent definition of the Mañé strict critical value (1.2) (cf. [4])

$$c_0(L) = \min\big\{\alpha([\omega]) \colon [\omega] \in H^1(\mathbb{T}^2, \mathbb{R})\big\} = -\beta(0).$$

In [1], M. J. Carneiro proved that the set $\widetilde{\mathcal{M}}_L([\omega])$ is contained in the energy level $E_L^{-1}(\alpha([\omega]))$ and, by the above characterizations (see for example [4]), we have that

$$\widetilde{\mathcal{M}}_L([\omega]) \subseteq \widetilde{\mathcal{A}}_L([\omega]) \subseteq \widetilde{\mathcal{N}}_L([\omega]) \subseteq E^{-1}(\alpha([\omega])).$$

By the graph property, we can define an equivalence relation in the set $\widetilde{\mathcal{A}}_L([\omega])$: two elements θ_1 and $\theta_2 \in \widetilde{\mathcal{A}}_L([\omega])$ are equivalent when $\delta_{[\omega]}(\pi(\theta_1), \pi(\theta_2)) = 0$. The equivalence relation breaks $\widetilde{\mathcal{A}}_L([\omega])$ down into classes that are called *static classes of* L. Let $\mathbf{\Lambda}_L([\omega])$ be the set of all static classes. We define a partial order $\preceq$ in $\mathbf{\Lambda}_L([\omega])$ by: (i) $\preceq$ is reflexive and transitive, (ii) if there is $\theta \in \widetilde{\mathcal{N}}_L([\omega])$, such that the α-limit set $\alpha(\theta) \subset \Lambda_i$ and the ω-limit set $\omega(\theta) \subset \Lambda_j$, then $\Lambda_i \preceq \Lambda_j$. The following theorem was proved by G. Contreras and G. Paternain in [5].

Theorem 2. *Suppose that the number of static classes is finite. Then, given* Λ_i *and* Λ_j *in* $\mathbf{\Lambda}_L([\omega])$, *we have that* $\Lambda_i \preceq \Lambda_j$.

Let $\Gamma \subset T\mathbb{T}^2$ be an invariant subset. Given $\epsilon > 0$ and $T > 0$, we say that two points $\theta_1, \theta_2 \in \Gamma$ are (ϵ, T)-*connected by chain* in Γ if there is a finite sequence $\{(\xi_i, t_i)\}_{i=1}^n \subset \Gamma \times \mathbb{R}$, such that $\xi_1 = \theta_1$, $\xi_n = \theta_2$, $T < t_i$ and $\operatorname{dist}(\phi_{t_i}(\xi_1), \xi_{i+1}) < \epsilon$, for $i = 1, \ldots, n-1$. We say that the subset Γ is *chain transitive* if for all $\theta_1, \theta_2 \in \Gamma$ and for all $\epsilon > 0$ and $T > 0$ the points θ_1 and θ_2 are (ϵ, T)-connected by chain in Γ. When this condition holds for $\theta_1 = \theta_2$, we say that Γ is *chain-recurrent.* The proof of the following properties can be seen in [2].

Theorem 3. $\widetilde{\mathcal{A}}_L([\omega])$ *is chain-recurrent.*

The following theorem was proved by Mañé in [8]. A proof can be seen also in [2, Theorem IV].

Theorem 4. *Let* $\mu \in \mathfrak{B}(L)$*. Then* $\mu \in \mathfrak{M}_L([\omega])$ *if only if* $Supp(\mu) \subset \widetilde{\mathcal{A}}_L([\omega])$*.*

Finally, for any closed manifold M, we say that a class $h \in H_1(M, \mathbb{R})$ is a *rational homology* if there is $\lambda > 0$ such that $\lambda h \in i_* H_1(M, \mathbb{Z})$, where $i\colon H_1(M, \mathbb{Z}) \hookrightarrow H_1(M, \mathbb{R})$ denotes the inclusion. The following proposition was proved by D. Massart in [9].

Proposition 1. *Let* M *be a closed and oriented surface and let* L *be a Tonelli Lagrangian on* M*. Let* $\mu \in \mathfrak{B}(L)$ *be a measure* $\rho(\mu)$*-minimizing such that* $\rho(\mu)$ *is a rational homology. Then the support of* μ *is a union of closed orbits or fixed points of the Lagrangian flow.*

3. Proof of Theorem 1

In this section, we prove Theorem 1.

Let $L\colon T\mathbb{T}^2 \to \mathbb{R}$ and $c > c_0(L)$. We assume that the restricted flow

$$\phi_t|_{E_L^{-1}(c)} : E_L^{-1}(c) \to E_L^{-1}(c)$$

satisfies two conditions:

(c1) all closed orbits are hyperbolic or elliptic, and

(c2) all heteroclinic intersections are transverse.

Lemma 1. *Let* $c > c_0(L)$ *and let* $h_0 \in H_1(\mathbb{T}^2, \mathbb{R}) \approx \mathbb{R}^2$ *be a nonzero class. Then there are an invariant probability measure* μ_0 *and a closed 1-form* ω_0 *with* $\alpha([\omega_0]) = c$*, such that:*

(i) $\rho(\mu_0) = \lambda_0 h_0$ *for some* $\lambda_0 \in R$*,*

(ii) $\mu_0 \in \mathfrak{M}_L([\omega_0])$ *and therefore* $Supp(\mu_0) \subset E_L^{-1}(c)$*.*

Proof. Since β is superlinear, we have

$$\lim_{\lambda \to \infty} \frac{\beta(\lambda h_0)}{|\lambda h_0|} = \infty. \tag{3.1}$$

Let $\partial\beta\colon H_1(\mathbb{T}^2, \mathbb{R}) \to H_1(\mathbb{T}^2, \mathbb{R})^* = H^1(\mathbb{T}^2, \mathbb{R})$ be the multivalued function such that to each point $h \in H_1(\mathbb{T}^2, \mathbb{R})$ associates all subderivatives of β at the point h. It is well known that, since β is finite, $\partial\beta(h)$ is a nonempty convex cone for all $h \in H_1(\mathbb{T}^2, \mathbb{R})$, and $\partial\beta(h)$ is a unique vector if and only if β is differentiable in h (see, for example, [14, Section 23]). We define the subset

$$S(h_0) = \bigcup_{\lambda \in \mathbb{R}} \partial\beta(\lambda h_0).$$

By (3.1) we have that the subset $S(h_0) \subset H^1(\mathbb{T}^2, \mathbb{R})$ is not bounded. Since β is continuous, by the above properties of the multivalued function $\partial\beta$, we have that $S(h_0)$ is a convex subset. Observe that, if $\omega \in \partial\beta(0)$, then $\alpha([\omega]) = c_0(L) = \min\{\alpha([\delta]);\, \delta \in H^1(\mathbb{T}^2, \mathbb{R})\}$ and, by superlinearity of α, the restriction $\alpha|_{S(h_0)}$ is not bounded. Therefore, by the intermediate value theorem, for each $c \in [c_0(L), \infty)$ there is $\omega_0 \in \partial\beta(\lambda_0 h_0) \subset S(h_0)$, for some $\lambda_0 \in \mathbb{R}$, such that $\alpha([\omega_0]) = c$. Therefore, if $\mu_0 \in \mathfrak{M}(L)$ is a $(\lambda_0 h_0)$-minimizing measure, then $\mu_0 \in \mathfrak{M}_L([\omega_0])$, and $Supp(\mu_0) \subset E_L^{-1}(c)$ (by [1]). $\square$

Let $i\colon H_1(\mathbb{T}^2,\mathbb{Z}) \hookrightarrow H_1(\mathbb{T}^2,\mathbb{R})$ be the inclusion. Recall that $H^1(\mathbb{T}^2,\mathbb{Z}) \approx \mathbb{Z}^2$ and that $H^1(\mathbb{T}^2,\mathbb{R}) \approx \mathbb{R}^2$. Then $\{(0,1),\,(1,0)\} \subset H_1(\mathbb{T}^2,\mathbb{Z})$ is a base of $H_1(\mathbb{T}^2,\mathbb{Z})$. We have that, if α_0, α_1 are two closed curves in $\mathbb{T}^2$, with $[\alpha_0] = (0,1)$ and $[\alpha_1] = (1,0)$, then $\alpha_0 \cap \alpha_1 \neq \varnothing$.

We fix $h_0 = (0,1) \in H_1(\mathbb{T}^2,\mathbb{Z})$. By applying Lemma 1 we obtain a closed 1-form ω_0 and a $[\omega_0]$-minimizing measure μ_0 with support into the level $E_L^{-1}(c)$, for which the rotational vector $\rho(\mu_0) = \lambda_0 h_0$ is a rational homology class. Therefore, by Proposition 1, the support of μ_0 is formed by the union of closed orbits of the flow $\phi_t|_{E_L^{-1}(c)}$.

Lemma 2. *The Mather set $\widetilde{\mathcal{M}}_L([\omega_0])$ is the union of a finite number of closed orbits for the Lagrangian flow of L.*

Proof. The Mather graph theorem asserts that the map $\pi|_{\widetilde{\mathcal{M}}_L([\omega_0])}\colon \widetilde{\mathcal{M}}_L([\omega_0]) \to \mathbb{T}^2$ is injective. Hence, if θ_1, $\theta_2\colon \mathbb{R} \to \mathbb{T}^2$ are two distinct closed orbits of ϕ_t contained in $\widetilde{\mathcal{M}}_L([\omega_0])$, then $\gamma_1(t) = \pi\circ\theta_1(t)$ and $\gamma_2(t) = \pi\circ\theta_2(t)$ must be simple closed curves and $[\gamma_1] = n[\gamma_2] \in H_1(\mathbb{T}^2,\mathbb{Z})$, because otherwise $\gamma_1 \cap \gamma_2 \neq \varnothing$.

Since $c = \alpha([\omega_0]) > c_0(L) = -\beta(0)$ and $\mathfrak{M}_L([\omega_0])$ is a compact set, the continuity of the map $\rho\colon \mathfrak{B}(L) \to H_1(\mathbb{T}^2,\mathbb{R})$ (c.f. [10]) implies that there are constants k, $l \in \mathbb{R}$ such that

$$0 < k \leqslant |\rho(\mu)| \leqslant l, \qquad \text{for all } \mu \in \mathfrak{M}_L([\omega_0]). \tag{3.2}$$

By the definition of $\widetilde{\mathcal{M}}_L([\omega_0])$, we have that $Supp(\mu_0) \subset \widetilde{\mathcal{M}}_L([\omega_0])$. Let μ_γ be the ergodic measure supported in a closed orbit $\theta(t) = (\gamma(t),\dot\gamma(t)) \subset Supp(\mu_0)$. Since $\rho(\mu_0) = \lambda_0 h_0$ and by the linearity of the map ρ, we have that $[\gamma] = n_0 h_0$ for some $0 \neq n_0 \in \mathbb{Z}$. It follows from (2.1) that

$$\langle \rho(\mu_\gamma), [\omega]\rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t \omega(\theta(s))\,ds = \frac{1}{|n_0|T}\int_0^{|n_0|T} \omega(\theta(s))\,ds = \frac{1}{|n_0|T}\int_\gamma \omega,$$

for any closed 1-form ω on $\mathbb{T}^2$. Then

$$\rho(\mu_\gamma) = \frac{[\gamma]}{|n_0|T} = \pm\,\frac{h_0}{T}, \tag{3.3}$$

where $T > 0$ denotes the minimal period of γ. Therefore, the period of any periodic orbit contained in $Supp(\mu_0)$ is bounded.

We fix a simple closed orbit $\gamma \in \pi(Supp(\mu_0))$. Since $[\gamma] = n_0 h_0$, we have that the set $C_\gamma = \mathbb{T}^2 - \{\gamma\}$ defines an open cylinder. Let $\mu \neq \mu_0$ be contained in $\mathfrak{M}_L([\omega_0])$. The graph property implies that $Supp(\mu) \cap Supp(\mu_0) = \varnothing$. Then $\pi(Supp(\mu)) \subset C_\gamma$ and $\rho(\mu) \in i_*(H_1(C_\gamma,\mathbb{R})) \subset H_1(\mathbb{T}^2,\mathbb{R})$. Therefore, we have that

$$\mathfrak{M}_L([\omega_0]) \subset \{\mu \in \mathfrak{M}(L)\colon \rho(\mu) \in \langle h_0\rangle_{\mathbb{R}})\}.$$

Applying Proposition 1 in each measure of the set $\mathfrak{M}_L([\omega_0])$, we conclude that the set $\widetilde{\mathcal{M}}_L([\omega_0])$ is a union of closed orbits for the Lagrangian flow. Therefore, the same arguments used on the ergodic components of μ_0 imply that each ergodic measure in $\widetilde{\mathcal{M}}_L([\omega_0])$ satisfies equation (3.3). Then inequality (3.2) implies that the period of all periodic orbits in $\widetilde{\mathcal{M}}_L([\omega_0])$ is uniformly bounded. Therefore, it follows from the compactness of $E_L^{-1}(c)$ and condition (c1) that there is at most a finite number of closed orbits of $\phi_t|_{E_L^{-1}(c)}$ in the Mather set $\widetilde{\mathcal{M}}_L([\omega_0])$. □

REMARK 1. It is well known that action minimizing curves do not contain *conjugate points*[2] and a proof of this fact can be seen in [3, § 4]. So, by Proposition A in [3], for each $\theta \in \widetilde{\mathcal{M}}_L([\omega_0])$ there exists the Green bundle along $\phi_t(\theta)$, i.e., there is a ϕ_t-invariant bi-dimensional subspace $\mathbb{E}(\phi_t(\theta)) \subset T_{\phi_t(\theta)}E^{-1}(c) \cong \mathbb{R}^3$ for all $t \in \mathbb{R}$. Therefore, the linearized Poincaré map on θ has an invariant one-dimensional subspace, so the periodic orbit $\phi_t(\theta)$ cannot be elliptic.

By Lemma 2, let $\gamma_i \colon \mathbb{R} \to \mathbb{T}^2$, with $i = 1, \ldots, n$, be a closed curves such that

$$\mathcal{M}_L([\omega_0]) = \bigcup_{i=1}^{n} \gamma_i.$$

Since $Supp(\mu_0) \subset \widetilde{\mathcal{M}}_L([\omega_0])$, we have that $[\gamma_i] = n_0 h_0 = (0, n_0) \in H_1(\mathbb{T}^2, \mathbb{Z})$, for all $i \in \{1, \ldots, n\}$.

Let $\widetilde{\mathcal{A}}_L([\omega_0])$ be the Aubry set corresponding to the cohomology class $[\omega_0]$ and let $\mathbf{\Lambda}_L([\omega_0])$ be the set of all static classes. We recall that

$$\widetilde{\mathcal{M}}_L([\omega_0]) \subseteq \widetilde{\mathcal{A}}_L([\omega_0]).$$

We have that either $\widetilde{\mathcal{M}}_L([\omega_0]) \neq \widetilde{\mathcal{A}}_L([\omega_0])$ or $\widetilde{\mathcal{M}}_L([\omega_0]) = \widetilde{\mathcal{A}}_L([\omega_0])$.

If $\widetilde{\mathcal{M}}_L([\omega_0]) \neq \widetilde{\mathcal{A}}_L([\omega_0])$, for each $\theta \in \widetilde{\mathcal{A}}_L([\omega_0]) \backslash \widetilde{\mathcal{M}}_L([\omega_0])$, by the graph property of the Aubry set $\widetilde{\mathcal{A}}_L([\omega_0])$, the curve $\gamma_\theta := \pi \circ \phi_t(\theta) \colon \mathbb{R} \to \mathbb{T}^2$ has no self-intersection points and $\gamma_\theta \cap \mathcal{M}_L([\omega_0]) = \varnothing$. Moreover, by Theorem 4, we have that the α-limit and ω-limit sets of θ are contained in the Mather set $\widetilde{\mathcal{M}}_L([\omega_0])$. Since a curve on $\mathbb{T}^2$ that accumulates in positive time to more than one closed curve must have self-intersection points, we have that $\omega(\theta)$ is a single closed orbit. By the same arguments, we have that $\alpha(\theta)$ is a single closed orbit. By Remark 1 and condition (c2), the orbit $\phi_t(\theta)$ is in the transverse intersections of an unstable manifold with a stable manifold of hyperbolic closed orbits, i.e., $\phi_t(\theta)$ is a transverse heteroclinic orbit. Certainly, if $\widetilde{\mathcal{M}}_L([\omega_0])$ is a unique closed orbit, then $\phi_t(\theta)$ is a transverse homoclinic orbit, which implies $h_{top}(L, c) > 0$ (see, for example, [7, p. 276]). If the set $\widetilde{\mathcal{M}}_L([\omega_0]$ has more than one orbit, it follows from the recurrence property (Theorem 3) that θ is an (ϵ, T)-chain connected in $\widetilde{\mathcal{A}}_L([\omega_0])$ for all $\epsilon > 0$ and $T > 0$, i.e., there is a finite sequence $\{(\zeta_i, t_i)\}_{i=1}^{k} \subset \widetilde{\mathcal{A}}_L([\omega_0]) \times \mathbb{R}$, such that $\zeta_1 = \zeta_k = \theta$, $T < t_i$ and $\operatorname{dist}(\phi_{t_i}(\zeta_1), \zeta_{i+1}) < \epsilon$, for $i = 1, \ldots, k-1$. Since the closed orbits in $\widetilde{\mathcal{M}}_L([\omega_0])$ are isolated on the torus, we have that for ϵ small enough the set $\{\pi(\zeta_i)\}_{i=1}^{k} \subset \mathcal{A}_L([\omega_0])$ must intersect the interior of each of the cylinders obtained by cutting the torus along the two curves $\gamma_i, \gamma_j \in \mathcal{A}_L([\omega_0])$, with $1 \leqslant i,\ j \leqslant n$. Therefore, choosing an orientation on $\mathcal{A}_L([\omega_0])$ and reordering the indices, we obtain a cycle of transverse heteroclinic orbits. This implies that $h_{top}(L, c) > 0$.

Now we consider the case of $\widetilde{\mathcal{M}}_L([\omega_0]) = \widetilde{\mathcal{A}}_L([\omega_0])$. Since each static class is a connected set (Proposition 3.4 in [5]), for each $1 \leqslant i \leqslant n$, we have that $\Lambda_i = (\gamma_i, \dot{\gamma}_i)$. Hence, the number of static classes is equal to n.

Initially, let us assume that $\widetilde{\mathcal{A}}_L([\omega_0]$ has at least two static classes, i.e., $n \geqslant 2$. Let Λ_1 and Λ_2 be two distinct static classes. Applying Theorem 2, we have that $\Lambda_1 \preceq \Lambda_2$, therefore, there exist classes $\Lambda_1, \ldots, \Lambda_j = \Lambda_2$ and points $\theta_1, \ldots, \theta_{j-1} \in \widetilde{\mathcal{N}}_L([\omega_0])$ such that, for all $1 \leqslant i \leqslant j-1$, the α-limit set $\alpha(\theta_i) \subset \Lambda_i$ and the ω-limit set $\omega(\theta_i) \subset \Lambda_{i+1}$. Also, we have that $\Lambda_2 \preceq \Lambda_1$, then there exist $\Lambda_{j+1}, \ldots, \Lambda_{j+k} = \Lambda_1$ and points $\theta_{j+1}, \ldots, \theta_{j+k-1} \in \widetilde{\mathcal{N}}_L([\omega_0])$ such that, for

[2] Two points $\theta_1 \neq \theta_2$ are said to be conjugate if $\theta_2 = \phi_\tau(\theta_2)$ and $V(\theta_2) \cap d_{\theta_1}\phi_\tau(V(\theta_1)) \neq \{0\}$, where $V(\theta) = \ker d_\theta \pi$ denotes the vertical bundle.

all $j+1 \leqslant i \leqslant j+k-1$, we have that $\alpha(\theta_i) \subset \Lambda_i$ and $\omega(\theta_i) \subset \Lambda_{i+1}$. By Remark 1, the corresponding orbits $(\gamma_i, \dot{\gamma}_i) = \Lambda_i$ for $1 \leqslant i \leqslant j+k$ are hyperbolic closed orbits of the flow $\phi_t|_{E_L^{-1}(c)}$, and since the flow $\phi_t|_{E^{-1}(c)}$ satisfies condition (c2), we have that for all $1 \leqslant i \leqslant j+k$ the orbits $\phi_t(\theta_i)$ are in the transverse heteroclinic intersections of the unstable manifold of Λ_i and the stable manifold of Λ_{i+1}. Then we obtain a cycle of transverse heteroclinic orbits, in particular, we have that $h_{top}(L, c) > 0$.

Let us assume now that $\widetilde{\mathcal{A}}_L([\omega_0])$ contains only one static class $\Lambda_1 = (\gamma_1, \dot{\gamma}_1)$. In this case, we can apply Theorem B in [5], which implies that the stable and unstable manifolds of the hyperbolic closed orbit Λ_1 have transverse homoclinic intersections. Here we will give a proof of this result in our particular setting.

Let $\Gamma = 2\mathbb{Z} \times \mathbb{Z}$ be a sublattice of $\mathbb{Z}^2 \subset \mathbb{R}^2$. We denote by $\overline{\mathbb{T}^2}$ the torus defined using the sublattice Γ, that is, $\overline{\mathbb{T}^2} = \mathbb{R}^2/\Gamma$. Let $p\colon \overline{\mathbb{T}^2} \to \mathbb{T}^2$ be the canonical projection. Note that $p\colon \overline{\mathbb{T}^2} \to \mathbb{T}^2$ is a double-covering of $\mathbb{T}^2$. Let $\overline{L}\colon \overline{\mathbb{T}^2} \to \mathbb{R}$ be the lift of the Lagrangian L to $\overline{\mathbb{T}^2}$. It is well known that $c_0(\overline{L}) = c_0(L)$ and we have that

$$\mathcal{A}_{\overline{L}}([\omega_0]) = p^{-1}(\mathcal{A}_L([\omega_0])) = p^{-1}(\Lambda_1)$$

(cf. [5, Lemma 2.3]). By the construction of the covering $p\colon \overline{\mathbb{T}^2} \to \mathbb{T}^2$, the set $p^{-1}(\Lambda_1) \subset \overline{\mathbb{T}^2}$ has two connected components. So, the Aubry set $\widetilde{\mathcal{A}}_{\overline{L}}([\omega_0])$ has two static classes and, as in the above case, there is a heteroclinic orbit connecting these classes. This heteroclinic connection is projected in a transverse homoclinic orbit of the hyperbolic orbit $\Lambda_1 = (\gamma_1, \dot{\gamma}_1) \subset \mathbb{T}^2$. This completes the proof.

We note that the homoclinic orbit that we obtained is not in the Mañé set $\widetilde{\mathcal{N}}_L([\omega_0])$, however, it is a projection of an orbit in the Mañé set for the corresponding lifted Lagrangian to the covering $p\colon \overline{\mathbb{T}^2} \to \mathbb{T}^2$.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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