# Weak Values and the quantum phase space

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We address the issue of how to properly treat, and in a more general setting, the concept of a weak value of a weak measurement in quantum mechanics. We show that for this purpose, one must take in account the effects of the measuring process on the entire phase space of the measuring system. By using coherent states, we go a step further than Jozsa in a recent paper, and we present an example where the result of the measurement is symmetrical in the position and momentum observables and seems to be much better suited for quantum optical implementation.

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## I. INTRODUCTION

The concept of a *weak value* of a quantum mechanical system was introduced in 1988 by Aharonov, Albert and Vaidman [1, 2]. It was built on a *time symmetrical model* for quantum mechanics previously introduced by Aharonov, Bergmann and Lebowitz in 1964 [3]. In this model, non-local time boundary conditions are used, since the description of the state of a physical system between two quantum mechanical measurements is made by pre and post-selection of the states. The authors developed the so called *ABL Rule* for the transition probabilities within this scenario, so this is why it is also known as the *two state formalism* for quantum mechanics [4]. The *weak value* of an observable can be considered as a generalization of the usual expectation value of a quantum observable, but differently from this, it takes values in the *complex plane* in general. Recently, Jozsa presented a deeper understanding of the physical meaning of the *real* and *imaginary* part of the complex value [5]. In this letter, we review some of Jozsa's ideas and suggest further progress in the comprehension of the weak value through a more general analysis of the effect of the weak measuring process on the quantum phase plane of the measuring system. Every measurement (whether a common or a weak measurement) can be understood as an interaction of the system being measured with the measuring system (the "measuring device") described by the von-Neumann model [6]. In the limit of an infinitesimally small coupling of the measuring system to the system to be measured, the first system will be accordingly "infinitesimally perturbed", but this small effect can be revealed by taking an average over the measurements of a very large ensemble of identically prepared systems with the same pair of pre and post selected states. We propose a more general analysis of the physical meaning of a weak value than those found in current literature (to the extent of our knowledge) through a quantum phase space description of the measuring system. In sections II and III, we briefly review the concept of the ideal von-Neumann measurement model and the quantum phase space formalism through coherent states. In section IV, we review the weak value concept within the quantum phase space formalism and state the main result of this work. In section V, we address some concluding remarks and suggestions for further research.

## II. THE VON-NEUMANN MODEL FOR AN IDEAL MEASUREMENT

Let $W = W_S \otimes W_M$ be the state vector space of the system formed by the subsystem $W_S$ and the *measuring* subsystem $W_M$. Assume further, that we are interested in measuring a discrete quantum variable of $W_M$ defined by the observable $\widehat{O} = \sum_i |o_i\rangle o_i \langle o^i|$ and that the measuring subsystem, for simplification purposes, will be considered as a structureless (no spin or internal variables) quantum mechanical particle in one dimension. Thus, we can choose as a basis for the vector state space $W_M$ either of the usual eigenstates of position or momentum $\{|q(x)\rangle\}$ or $\{|p(x)\rangle\}$. It is important to note here that we use a slightly different notation than usual (for reasons that will soon become evident) in the sense that we distinguish between the "type" of the eigenvector ($q$ or $p$) from the actual $x$ eigenvalue

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[7]. For instance, we write:

$$\widehat{Q}|q(x)\rangle = x|q(x)\rangle \qquad \text{and} \qquad \widehat{P}|p(x)\rangle = x|p(x)\rangle, \tag{1}$$

(instead of $\widehat{Q}|q\rangle = q|q\rangle$ and $\widehat{P}|p\rangle = p|p\rangle$ as commonly written) where $\widehat{Q}$ and $\widehat{P}$ are the position and momentum observables subject to the well known Heisenberg relation: $[\widehat{Q}, \widehat{P}] = i\widehat{I}$ (hereinafter, $\hbar = 1$ units will be used). With this non-standard notation, the completeness relation and the overlapping between these bases can be written as:

$$\int_{-\infty}^{+\infty} |q(x)\rangle\langle q(x)|dx = \int_{-\infty}^{+\infty} |p(x)\rangle\langle p(x)|dx = \widehat{I} \quad \text{and} \quad \langle q(x)|p(x\prime)\rangle = \frac{e^{ixx\prime}}{\sqrt{2\pi}}. \tag{2}$$

An *ideal* von-Neumann measurement can be defined as an *instantaneous* interaction between the two subsystems as modeled by the following Dirac delta-like time-pulse hamiltonian operator at time $t_0$:

$$\widehat{H}_{int}(t) = \lambda\delta(t - t_0)\widehat{O} \otimes \widehat{P}, \tag{3}$$

where $\lambda$ is a parameter that represents the intensity of the interaction. This ideal situation models a setup where we are supposing that the time of interaction is very small compared to the time evolution given by the free Hamiltonians of both subsystems.

Let the initial state of the total system be given by the following uncorrelated product state: $|\psi_{(i)}\rangle = |\alpha\rangle \otimes |\varphi_{(i)}\rangle$ and the final state given by $|\psi_{(f)}\rangle = \widehat{U}(t_A, t_B)|\psi_{(i)}\rangle \quad (t_A < t_0 < t_B)$, where the total unitary evolution operator is

$$\widehat{U}(t_A, t_B) = e^{-i\int_{t_A}^{t_B} \widehat{H}_{int}(t)dt} = e^{-i\lambda\widehat{O}\otimes\widehat{P}}, \tag{4}$$

such that

$$(\widehat{I} \otimes \langle q(x)|)|\psi_{(f)}\rangle = \sum_j |o_j\rangle \otimes \langle q(x)|\widehat{V}^{\dagger}_{o_j}|\varphi_{(i)}\rangle\alpha^j, \tag{5}$$

where $|\alpha\rangle = \sum_j |o_j\rangle\langle o^j|\alpha\rangle = \sum_j |o_j\rangle\alpha^j$ and $\widehat{V}_\xi$ is the one-parameter family of unitary operators in $W_M$ that represents the *Abelian group of translations* in the position basis ($x \in \Re$):

$$\widehat{V}_\xi|q(x)\rangle = |q(x - \xi)\rangle \tag{6}$$

A correlation in the final state of the total system is then established between the variable to be measured $o_j$ with the continuous position variable of the measuring particle:

$$(\widehat{I} \otimes \langle q(x)|)|\psi_{(f)}\rangle = \sum_j |o_j\rangle\alpha^j\varphi_{(i)}(x - \lambda o_j), \tag{7}$$

where $\varphi_{(i)}(x) = \langle q(x)|\varphi_{(i)}\rangle$ is the wave-function in the position basis of the measuring system (the 1-D particle) in its initial state. This step of the von-Neumann measurement prescription is called the *pre-measurement* of the system. The true measurement happens effectively when an observation of the measuring system (which is considered to be "classical" in some sense) is carried out. If this is the case, one obtains an observed value $x - \lambda o_j$ of the position of the classical particle with a probability $P_j = |\alpha_j|^2$. In the Copenhagen interpretation of quantum physics, the existence of a "classical description" of reality is *aprioristically* necessary for the description of quantum reality. This kind of pragmatic and extreme positivistic position, advocated by many physicists (Bohr and Landau just to mention some of the most prominent) has always been the center of a heated debate from the very beginning of quantum theory [8]. In the last twenty years, a program that tries to offer a solution to this problem through a full quantum description of the measurement process making use of the concept of *decoherence* (the inevitable entangling between the measuring system and the environment) has emerged resulting in a number of important achievements in various aspects of the theory. In this work, we are not concerned with these difficult and foundational aspects of the measurement process. But we should mention that a full and completely agreed upon resolution of the *measurement problem* in quantum mechanics does not seem to have been proposed yet [9].

## III. THE QUANTUM PHASE SPACE

By a *quantum symplectic transform*, we mean a unitary transformation in $W_M$ that implements a representation of the group of area preserving linear transformations of the classical phase plane. For instance, the usual Fourier transform operator $\widehat{F}$ represents a $\pi/2$ rotation of the phase plane. In fact, given a coherent state $\mid p, q\rangle$ representing a point in phase plane, one can show that $\widehat{F} \mid p, q\rangle = \mid q, -p\rangle$ [7]. In 1980, Namias developed the concept of a *fractionary Fourier transform* which has been used since then in various applications in optics, signal processing and quantum mechanics [10]. This operator is nothing else but an arbitrary Euclidean rotation in the phase plane. By *Euclidean*, we mean a linear transformation that *preserves* the usual metric in $\Re^2$ with positive determinant, that is, the one-parameter Abelian group $SO(2)$. This, of course, does not exhaust all area preserving linear transformations of the plane which is the non-Abelian $SL(2, \Re)$ group. We can define the Fourier transform operator as

$$\widehat{F} = \int_{-\infty}^{+\infty} dx \mid p(x)\rangle\langle q(x)|. \tag{8}$$

(Note that it would be impossible to define the Fourier operator in such a clean and direct way with the usual $\mid q\rangle$ and $\mid p\rangle$ notation for the position and momentum eigenstates). The squared Fourier operator $\widehat{F}^2$ is the space inversion operator and it can be shown that $\widehat{F}^3 = \widehat{F}^\dagger$ and $\widehat{F}^4 = \widehat{I}$, so it can be clearly seen that the eigenvalues of $\widehat{F}$ are the fourth roots of unity. In fact, it is well known that

$$\widehat{F} \mid n\rangle = (i)^n \mid n\rangle, \tag{9}$$

where $\{\mid n\rangle\}$, $(n = 0, 1, 2 \ldots)$ is the complete set of eigenkets of the *number operator* $\widehat{N} = \widehat{a}^t\widehat{a}$, which is in itself, the generator of rotations in the phase plane [7]. Probably the best way to visualize this is through the identification of the phase plane with the complex plane via the standard complex-valued coherent states defined by the following change of variables:

$$z = \frac{1}{\sqrt{2}}(q + ip) \tag{10}$$

and defined as

$$\mid z\rangle = \widehat{D}[z] \mid 0\rangle, \tag{11}$$

with the displacement $\widehat{D}[z]$ operator given by

$$\widehat{D}[z] = e^{(z\widehat{a}^t - \bar{z}\widehat{a})}. \tag{12}$$

It is not difficult then to show that indeed

$$e^{i\theta\widehat{N}} \mid z\rangle = \mid e^{i\theta} z\rangle. \tag{13}$$

so that,

$$\widehat{F}_\theta = e^{i\theta\widehat{N}} \tag{14}$$

is in fact the Namias fractional Fourier operator and the usual Fourier operator is the special case with $\theta = \pi/2$. We can rewrite $\widehat{N}$ in terms of the position and momentum operators since $\widehat{a} = \frac{1}{\sqrt{2}}(\widehat{Q} + i\widehat{P})$, so the number operator may be expressed as

$$\widehat{N} = \widehat{H}_0 - \frac{1}{2}\widehat{I} = \frac{1}{2}(\widehat{Q}^2 + \widehat{P}^2) - \frac{1}{2}\widehat{I}, \tag{15}$$

where $\widehat{H}_0$ is the Hamiltonian of a unit mass and unit frequency simple harmonic oscillator. Since the generator of rotations is *quadratic* in the canonical observables $\widehat{Q}$ and $\widehat{P}$, one may try to write down all possible quadratic operators in these variables: $\widehat{Q}^2$, $\widehat{P}^2$, $\widehat{Q}\widehat{P}$ and $\widehat{P}\widehat{Q}$, but the last two are obviously non-Hermitian so we could change them to the following (Hermitian) linear combinations: $\widehat{Q}\widehat{P} + \widehat{P}\widehat{Q}$ and $i(\widehat{Q}\widehat{P} - \widehat{P}\widehat{Q})$. The last one is proportional to the identity operator because of the Heisenberg commutation relation, so this leaves us with *three* linear independent operators that we choose as

$$\widehat{H}_0 = \frac{1}{2}(\widehat{Q}^2 + \widehat{P}^2) = \widehat{N} + \frac{1}{2}\widehat{I} = \widehat{a}^t\widehat{a} + \frac{1}{2}\widehat{I}, \tag{16}$$

$$\widehat{g} = \frac{1}{2}(\widehat{Q}\widehat{P} + \widehat{P}\widehat{Q}) = \frac{i}{2}[(\widehat{a}^t)^2 - \widehat{a}^2], \tag{17}$$

$$\widehat{k} = \frac{1}{2}(\widehat{Q}^2 - \widehat{P}^2) = \frac{1}{2}[(\widehat{a}^t)^2 + \widehat{a}^2]. \tag{18}$$

These three generators implement in $W_M$, the algebra $sl(2,\Re)$ of $SL(2,\Re)$. The $\widehat{g}$ operator is nothing but the squeezing generator from quantum optics [11]. The $\widehat{k}$ operator generates *hyperbolic rotations,* that is, linear transformations of the plane that preserve an *indefinite metric*. It takes the hyperbola $x^2 - y^2 = 1$ into itself in an analogous way that the Euclidean rotation takes the circle $x^2 + y^2 = 1$ into itself. $SL(2,\Re)$ is the Lie Group of all area preserving linear transformations of the plane, so we can identify it with the $2 \times 2$ real matrices with unit determinant. Since $\det e^X = e^{tr(X)}$, we can also identify its algebra $sl(2,\Re)$ with all $2\times 2$ real matrices with null trace. Thus, it is natural to make the following choice for a basis in this algebra:

$$\mathbf{X_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \widehat{\sigma}_1 \qquad \mathbf{X_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\widehat{\sigma}_2 \qquad \mathbf{X_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \widehat{\sigma}_3,$$

where we have written (for practical purposes) the elements of the algebra in terms of the well-known *Pauli matrices*. This is very adequate because physicists are familiar with the fact that the Pauli matrices $\{\frac{1}{2}\widehat{\sigma}_j\}$ form a two-dimensional representation of the angular momentum algebra:

$$[\widehat{\sigma}_i, \widehat{\sigma}_j] = 2i\widehat{\sigma}_k\epsilon^k_{ij}.$$

We can make use of these commutation relations to completely characterize the $sl(2,\Re)$ algebra. In fact, the mapping described by the table below relates these algebra elements directly to the algebra of their representation carried on $W_M$:

| **generators of** $sl(2,\Re)$ | **generators of the representation** |
|---|---|
| $X_1 \equiv \widehat{\sigma}_1$ | $-i\widehat{k}$ |
| $X_2 \equiv -i\widehat{\sigma}_2$ | $-i\widehat{H}_0$ |
| $X_3 \equiv \widehat{\sigma}_3$ | $-i\widehat{g}$ |

(19)

With a bit of work, it is not difficult to convince oneself that these mapped elements indeed obey identical commutation relations.

## IV. WEAK VALUES

The *weak value* of a quantum system introduced by Aharonov, Albert and Vaidman (A.A.V.) based on the two-state formalism for quantum mechanics generalizes the concept of an expectation value for a given observable. Let the *initial state* of the product space $W = W_S \otimes W_M$ be $|\psi_{(i)}\rangle = |\alpha\rangle \otimes |\varphi_{(i)}\rangle$ and a "*weak Hamiltonian*" given as:

$$\widehat{H}^{(w)}_{int}(t) = \epsilon\delta(t - t_0)\widehat{O} \otimes \widehat{P}, \qquad (\epsilon \to 0). \tag{20}$$

The *final state* will then be: $|\varphi_{(f)}\rangle = (\langle\beta|\otimes\widehat{I}).\widehat{U}^{(w)}(t_i,t_f).(|\alpha\rangle\otimes|\varphi_{(i)}\rangle)$, where $|\alpha\rangle$ and $|\beta\rangle$ are respectively the *pre and post-selected* states of the system to be measured $W_M$ and $\widehat{U}^{(w)}(t_i,t_f) \simeq \widehat{I} - i\epsilon\widehat{O}\otimes\widehat{P}$ is the *time evolution* operator for the weak interaction. In this way, we can compute to first order in $\epsilon$:

$$|\varphi_{(f)}\rangle \simeq \langle\beta|\alpha\rangle(1 - i\epsilon O_w\widehat{P})|\varphi_{(i)}\rangle \qquad \text{with} \qquad O_w = \frac{\langle\beta|\widehat{O}|\alpha\rangle}{\langle\beta|\alpha\rangle}. \tag{21}$$

Note that the weak value $O_w$ of the observable $\widehat{O}$ is, in general, an arbitrary *complex number*. Note also that, though $|\varphi_{(i)}\rangle$ is a normalized state, the $|\varphi_{(f)}\rangle$ state vector in general, *is not* normalized. In the original formulation of A.A.V, the momentum $\widehat{P}$ acts upon the measuring system, implementing a *small translation* of the initial wave function in the position basis, but which can be measured from the mean value of the results of a large series of identical experiments. That is, the expectation value of the position operator $\widehat{Q}$ over a large ensemble with the same pre and post selected states. Jozsa recently proposed a more general procedure by taking an arbitrary operator $\widehat{M}$ in the place of $\widehat{Q}$ as the observable of $W_M$ to be measured. In this case, the usual expectation values of $\widehat{M}$ in the initial and final states $|\varphi_{(i)}\rangle$ and $|\varphi_{(f)}\rangle$ are respectively:

$$\langle\widehat{M}\rangle_{(i)} = \langle\varphi_{(i)}|\widehat{M}|\varphi_{(i)}\rangle \qquad \text{and} \qquad \langle\widehat{M}\rangle_{(f)} = \frac{\langle\varphi_{(f)}|\widehat{M}|\varphi_{(f)}\rangle}{\langle\varphi_{(f)}|\varphi_{(f)}\rangle}. \tag{22}$$

Jozsa has shown that the difference between these expectation values to first order in $\epsilon$ is given by [5]: (the shift of $\langle\widehat{M}\rangle$ is defined as $\Delta\widehat{M} = \langle\widehat{M}\rangle_{(f)} - \langle\widehat{M}\rangle_{(i)}$)

$$\Delta\widehat{M} = \epsilon[(Im(O_w))(\langle\varphi_{(i)}|\{\widehat{M},\widehat{P}\}|\varphi_{(i)}\rangle - 2\langle\varphi_{(i)}|\widehat{P}|\varphi_{(i)}\rangle\langle\varphi_{(i)}|\widehat{M}|\varphi_{(i)}\rangle) - i(Re(O_w))\langle\varphi_{(i)}|[\widehat{M},\widehat{P}]|\varphi_{(i)}\rangle]. \tag{23}$$

He also discusses two different examples for $\widehat{M}$: At first, he chooses $\widehat{M} = \widehat{Q}$, so that (using the $sl(2,\Re)$ algebra and the Heisenberg commutation relation) he obtains:

$$\Delta\widehat{Q} = \epsilon[2(Im(O_w))(\langle\varphi_{(i)}|\widehat{g}|\varphi_{(i)}\rangle - \langle\varphi_{(i)}|\widehat{P}|\varphi_{(i)}\rangle\langle\varphi_{(i)}|\widehat{Q}|\varphi_{(i)}\rangle) + (Re(O_w))]. \tag{24}$$

By using the Heisenberg picture for time evolution and choosing the most general Hamiltonian for the measuring system $W_M$ and the relations of table 19.

$$\widehat{H}_M = \frac{1}{2m}\widehat{P}^2 + V(\widehat{Q}) \tag{25}$$

one obtains:

$$\frac{d\widehat{Q}}{dt} = \frac{\widehat{P}}{m} \qquad \text{and} \qquad \frac{d\widehat{Q}^2}{dt} = \frac{2}{m}\widehat{g}. \tag{26}$$

So one arrives at:

$$\Delta\widehat{Q} = \epsilon[(Re(O_w)) + m(Im(O_w))\frac{d}{dt}(\delta^2_{|\varphi_{(i)}\rangle}\widehat{Q})], \tag{27}$$

where $\delta^2_{|\varphi_{(i)}\rangle}\widehat{A} = \langle\varphi_{(i)}|\widehat{A}^2|\varphi_{(i)}\rangle - \langle\varphi_{(i)}|\widehat{A}|\varphi_{(i)}\rangle^2$ is the usual quadratic dispersion or uncertainty of the $\widehat{A}$ observable in an arbitrary state vector $|\varphi_{(i)}\rangle$. Analogously for $\widehat{M} = \widehat{P}$, comes:

$$\Delta\widehat{P} = 2\epsilon(Im(O_w))\delta^2_{|\varphi_{(i)}\rangle}\widehat{P}. \tag{28}$$

Note that there is a certain *asymmetry* in the results exhibited by the above equations. This is because of the *asymmetric* choice of the translation generator $\widehat{P}$ in the interaction Hamiltonian in equation 20. Note also that from equations 27 and 28 we can see that it is impossible to extract the *real* and *imaginary* values of $O_w$ with the measurement of $\Delta\widehat{Q}$ *only*, because both of these numbers are absorbed in a *same real* number. It is necessary to measure $\Delta\widehat{P}$ (besides knowing the values of $\frac{d}{dt}(\delta^2_{|\varphi_{(i)}\rangle}\widehat{Q})$ and $\delta^2_{|\varphi_{(i)}\rangle}\widehat{P}$). There is no reason why one should need to choose $\widehat{P}$ or $\widehat{Q}$ in the weak measurement Hamiltonian. We could choose any of the symplectic generators making use of the full symmetry of the $SL(2,\Re)$ group. The $\widehat{P}$ and $\widehat{Q}$ operators generate translations in phase space, but we can implement any area preserving transformation in the plane by also using observables that are quadratic in the momentum and position observables. We can also make use of our freedom of choice of an arbitrary initial state vector $|\varphi_{(i)}\rangle$ and the choice of an "adequate" Hamiltonian operator $\widehat{H}_M$ of the measuring system. We propose then a more general approach. Let us take an interaction Hamiltonian of the following form:

$$\widehat{H}^{(w)}_{int}(t) = \epsilon\delta(t - t_0)\widehat{O} \otimes \widehat{R} \qquad \text{with} \qquad (\epsilon \to 0), \tag{29}$$

where $\widehat{R}$ is any element of the algebra $sl(2,\Re)$, so it is the generator of an arbitrary symplectic transform of the measuring system. In this way, we can follow Jozsa's path obtaining the generalized $\Delta\widehat{M}$ shift in these conditions:

$$\Delta\widehat{M} = \epsilon[(Im(O_w))(\langle\varphi_{(i)}|\{\widehat{M},\widehat{R}\}|\varphi_{(i)}\rangle - 2\langle\varphi_{(i)}|\widehat{R}|\varphi_{(i)}\rangle\langle\varphi_{(i)}|\widehat{M}|\varphi_{(i)}\rangle) - i(Re(O_w))\langle\varphi_{(i)}|[\widehat{M},\widehat{R}]|\varphi_{(i)}\rangle]. \tag{30}$$

By choosing $\widehat{M} = \widehat{R}$, we get the analog of equation 28:

$$\Delta\widehat{R} = 2\epsilon(Im(O_w))\delta^2_{|\varphi_{(i)}\rangle}\widehat{R}. \tag{31}$$

For the second observable, we could choose any Hermitian operator that *does not commute* with $\widehat{R}$. This is because the main idea is to choose a "conjugate" variable to $\widehat{R}$ in a similar way that occurs with the $(\widehat{Q}, \widehat{P})$ pair. So a first obvious choice is to pick the number operator $\widehat{N}$ in the place of $\widehat{R}$. Since $\widehat{N}$ is the generator of Euclidean rotations in phase space, the annihilator operator $\widehat{a}$ seems a natural candidate choice to go along with $\widehat{N}$. Though $\widehat{a}$ is not Hermitean and as such, not a genuine observable, one may think that it would be useless as such. But any operator $\widehat{B}$ can be written as a sum of its hermitean and anti-hermitean components in the following manner:

$$\widehat{B} = \widehat{C} + i\widehat{D}, \tag{32}$$

where $\widehat{C}$ and $\widehat{D}$ are both Hermitean. In this manner we can define the expectation value of any operator in an arbitrary state vector $|\psi\rangle$ by [12].

$$\langle\widehat{B}\rangle_{|\psi\rangle} = \langle\widehat{C}\rangle_{|\psi\rangle} + i\langle\widehat{D}\rangle_{|\psi\rangle}. \tag{33}$$

Note also that by the linearity of $\widehat{M}$ in equation 23 we have

$$\Delta\widehat{B} = \Delta\widehat{C} + i\Delta\widehat{D}. \tag{34}$$

So we choose $(\widehat{N}, \widehat{a})$ in this manner as a candidate for a "conjugate pair" of operators as an analog to the pair $(\widehat{P}, \widehat{Q})$ because $\widehat{F}_\theta = e^{i\theta\widehat{N}}$ implements Euclidean rotations in phase plane while the coherent state $|z\rangle$ is an eigenket of $\widehat{a}$ in a similar way that the momentum operator implements translations in the position wave function. With this choice of $\widehat{M} = \widehat{a}$ it is not difficult to calculate the shift for the annihilator operator:

$$\Delta\widehat{a} = \epsilon[-iO_w\langle\varphi_{(i)}|\widehat{a}|\varphi_{(i)}\rangle + 2Im(O_w)(\langle\varphi_{(i)}|\widehat{N}\widehat{a}|\varphi_{(i)}\rangle - \langle\varphi_{(i)}|\widehat{N}|\varphi_{(i)}\rangle\langle\varphi_{(i)}|\widehat{a}|\varphi_{(i)}\rangle)]. \tag{35}$$

Unfortunately, the second term in the above equation cannot be identified with a "quadratic dispersion" for $\widehat{a}$ in the same way as Jozsa does for $\widehat{Q}$ in equation 27. In most models of weak measurements, the initial state of the measuring system is chosen to be a Gaussian state and the weak interaction promotes a small translation of its peak. But in a realistic quantum optical implementation of the measuring system, it is reasonable to choose the initial state of the system as a coherent state $|\varphi_{(i)}\rangle = |z\rangle$. In this case, there is a dramatic simplification for the shift:

$$\Delta\widehat{a} = -i\epsilon z O_w. \tag{36}$$

The above equation is the main result of this work. Note that equation 36 can be re-written as:

$$\Delta\widehat{a} = \epsilon\,|z|\,|O_w|\,e^{i(\theta_z+\theta_w-\pi/2)}, \tag{37}$$

where $z = |z|\,e^{i\theta_z}$ and $O_w = |O_w|\,e^{i\theta_w}$. If we make a convenient choice for the phase $\theta_z = \pi/2$ and use equation 34, we arrive at a symmetric pair of equations for $\Delta\widehat{Q}$ and $\Delta\widehat{P}$:

$$\Delta\widehat{Q} = \epsilon\sqrt{2}\,|z|\,Re(O_w) \tag{38}$$

and

$$\Delta\widehat{P} = \epsilon\sqrt{2}\,|z|\,Im(O_w). \tag{39}$$

These equations do not depend on the quadratic dispersion or the time derivative of the quadratic dispersion of any observable as it happens with the similar equation developed by Jozsa. An additional attractiveness of the above equations in comparison to those exhibited by Jozsa is the fact that one can in principle "tune" the size of the $\epsilon\,|z|$ term despite how small $\epsilon$ may be by making $|z|$ large enough. This may be of practical importance for optical implementation of weak value since $|z|$ for a quantized mode of an electromagnetic field is nothing else but the *mean photon number* in this mode in the $|z\rangle$ coherent state [11]. One may envisage an experiment with a drastic reduction of the size of the ensemble, maybe even measuring the weak value with *one single* experiment.

## V. CONCLUSIONS

The main message of our work is to underline the fact that the best way to understand the meaning of the complex weak value is to seek for the effects upon the phase space of the measuring system. We have shown an improvement of this understanding by choosing an experiment where the observable of the measuring system of the weak interaction Hamiltonian is chosen to be $\widehat{R} = \widehat{N}$ and the "variable" chosen to be measured is $\widehat{M} = \widehat{a}$. This seems a natural choice in a quantum optical experimental framework. Instead of looking to a translation of a Gaussian state as the usual proposals, we look to a "turn of the dial" of our coherent state $|z\rangle$ as a *pointer state*. It puts on a same basis the *real* and *complex* parts of the so called weak value in a natural way. We could indeed try to explore the full potentiality of all symplectic transforms in phase space, but he have chosen to start with the Euclidean rotation (or fractional Fourier transform) because of its very vivid and simple geometrical interpretation with the use of coherent states. We shall address, in a future paper, the full mathematical structure of all the symplectic group and its implementations [13]. Though the two state formalism of quantum mechanics appeared in the sixties decade of the last century, it is fair to say that it remained quite unnoticed until the concept of weak values was introduced by Aharonov, Albert and Vaidman in 1988. This novel idea has already shown a great deal of applications, but in our opinion the most interesting and important one is related to the phenomenon known as *quantum counterfactuality*. A widely known example is that of the *Elitzur-Vaidman bomb problem*, where a very sensitive photon detector (the "bomb") is itself "detected" by a photon without having really interacted with it. The fact that the paths of two distinct open quantum channels can interfere destructively (in the Feynman sense) allows the possibility of "detecting" a sensor which interrupts one of the channels in an interaction-free like experiment [14, 15]. Problems like these have been successfully analyzed through the concept of weak values, because these different paths may be actually "tested" without significantly perturbing the involved states. We believe that the understanding of this kind of approach to the study of quantum counterfactuality also could be enhanced by a shift to the general analysis of the effect on the phase space of the measuring systems that we are proposing. In this sense, it may be useful to consider discrete phase spaces as well [7]. In quantum optical applications, observables that are quadratic on the creation and annihilation operators are a routine matter, so we believe that this more general quantum phase space approach opens space for a gain of flexibility and a welcome increase of theoretical and experimental options for further investigations.

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