A note on a determinant identity

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Abstract

In this note we show that the determinant identities obtained by Rezaifar and Rezaee (2007)–[1] and Dutta and Pal (2011)–[2] are straightforward consequences of a general result due to Capelli.

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1. Introduction

In two interesting papers [1,2] the authors put forward an identity involving a representation for a determinant. Furthermore, they explore their approach in programming and establish a comparison with other standard existing methods (see, e.g., see section 7 of [1]). Here, our aim is to show that the results of Refs. [1,2] follow directly applying a well known general result due to Capelli (1855–1910) (see Ref. [3] and references therein or Eq. (1) below).

This note is organized as follows. In Section 2, we use the general formula of Capelli to recover the results of Refs. [1,2]. We follow closely Ref. [3]. See also Ref. [4]. In Section 3 we make some concluding remarks.

2. New proof of the determinant identity

Let us first give some simple definitions to fix notation. Let \( X = (x_{ij}) \) be an \( n \times n \) matrix. We take \( I, J \subseteq \{1, \ldots, n\} \) and \( I \cap J = \emptyset \). Also, we define \( \chi = \det X \), \( \partial_{ij} = (\partial / \partial x_{ij}) = (\partial_{ij})_{\epsilon \in I \cup J} \), \( \chi_{EF} = (x_{ij})_{\epsilon \in I \cup J} \), \( \chi_{EF} = \det \chi_{EF} \) and \( \epsilon(I, J) = (-1)^{\sum_{i \in I} \sum_{j \in J}} \). The following identity is generally attributed to Capelli (see Eq. (1.2) in Ref. [3]):

\[
\det (\partial_{ij}) \chi^s = s(s + 1) \ldots (s + k - 1) \chi^{s-1} \epsilon(I, J) \chi_{EF}.
\]

with \(|I| = k = |J|\). We refer the reader to Ref. [3] for further details towards the proof of this identity and a number of others generalizations, using methods from quantum field theory, like Grassmann–Berezin calculus. The identity in Eq. (1) is our starting point to show that

\[
\chi = \frac{1}{\chi_{(1, n)^c(1, n)^c}} \det \begin{pmatrix}
\chi_{(1)^c(1)^c} & \chi_{(1)^c(n)^c} \\
\chi_{(n)^c(1)^c} & \chi_{(n)^c(n)^c}
\end{pmatrix},
\]

which is equivalent to Eq. (3.1) of Ref. [1] by a suitable adjustment of notation. Indeed, let us take Eq. (1) with \( I = J = \{1, n\} \) (then \(|I| = |J| = k = 2\)) and \( s = 2 \). Therefore:

\[
\det (\partial_{(1, n)(1, n)}) \chi^2 = 6 \chi \epsilon(\{1, n\}, \{1, n\}) \chi_{(1, n)^c(1, n)^c},
\]
where we recall that
\[ \det \left( \theta_{1,n} \mid 1, n \right) = \det \left( \frac{\partial}{\partial n_1} - \frac{\partial}{\partial m_n} \right) \]
and \( X_{(i)^r(j)^s} \) is the determinant of the matrix \( (n - 1) \times (n - 1) \) obtained from \( X \) by deleting line \( i \) and column \( j \), the lines and columns \( 1 \) and \( n \), respectively. Now observe that \( \varepsilon((1, n), (1, n)) = (-1)^{2(n + 1)} = 1 \), therefore we can write for Eq. (3)
\[ \det \left( \theta_{1,n} \mid 1, n \right) \lambda^2 = 6 \lambda X_{r(1,n)^r} \lambda^2. \]
A direct calculation gives
\[ \det \left( \theta_{1,n} \mid 1, n \right) \lambda^2 = 2 \lambda \left[ \det \left( \theta_{1,n} \mid 1, n \right) \lambda^r \right] + 2 \left( \frac{\partial}{\partial n_1} \lambda X_{r n} - \frac{\partial}{\partial m_n} \lambda X_{r n} \right). \]
Now we apply Eq. (1) once again, this time with \( I = J = (1, n) \) (then \( |I| = |J| = k = 2 \)) and \( s = 1 \) to get
\[ \det \left( \theta_{1,n} \mid 1, n \right) \lambda = 2 \varepsilon((1, n), (1, n)) X_{r(1,n)^r} \lambda = 2 \lambda X_{r(1,n)^r} \lambda. \]
Now, taking Eqs. (5), (6) and going back to Eq. (4), observe that
\[ X \lambda X_{r(1,n)^r} = \frac{\partial}{\partial n_1} \lambda X_{r n} - \frac{\partial}{\partial m_n} \lambda X_{r n} \lambda. \]
Using, conveniently, the standard Laplace expansion of the determinant \( \lambda \), we can write
\[ \lambda = - \sum_j \left( -1 \right)^j \lambda_{[j]} \lambda X_{r(j)} = \lambda \left( -1 \right)^{1+n} \lambda X_{r(1,n)^r}, \]
\[ \lambda = \sum_j \left( -1 \right)^{1+n} \lambda_{1+j} \lambda X_{r(j)} \lambda X_{r(1,n)^r}. \]
Using the results above in Eq. (7) we have
\[ X \lambda X_{r(1,n)^r} = \lambda \left( -1 \right)^{1+n} \lambda X_{r(1,n)^r} = \lambda \left( -1 \right)^{1+n} \lambda X_{r(1,n)^r} \lambda X_{r(1,n)^r}, \]
and we get Eq. (2) of this note or, equivalently, Eq. (3.1) of Ref. [1]. It is clear from the procedure outlined in this section that the main result stated in Ref. [2], more precisely, Eq. (1) there, is also a consequence of Eq. (1) by taking \( I = J = (1, n) \) (note that \( \varepsilon((1, n), (1, n)) = 1 \)).

We close this section by noting that all the results stated here can be alternatively restated entirely in terms of the algebraic/combinatorial framework of Ref. [3]. We will limit ourselves to an indication of the main steps involved. Indeed, Eq. (8) follows from (4) and the following results of Ref. [3]. In what follows all the equations mentioned concern Ref. [3]. First, we take the Grassmann integral representation of \( \det(\widehat{h}_I)(\det X)^r \) in Eq. (5.13a) with \( s = 2 \) and \( I = (1, n) = J \) and use Eq. (4.1) to obtain a Grassmann-type representation for \( \det(X + \tilde{h} \eta^T) \), following the notation of Ref. [3]. Next, we use the properties of the Grassmann integral (see Eq. (A.56)) and we observe that the product \( \prod \tilde{h}_I \eta^T \) in the right-hand side of Eq. (5.13a) will select only certain elements of the matrix \( \tilde{h} \eta^T \) in the Grassmann (exponential) representation of \( \det(X + \tilde{h} \eta^T) \), i.e., \( \tilde{h}_1 \eta_1 \), \( \tilde{h}_1 \eta_2 \eta_3 \), \( \tilde{h}_1 \eta_2 \eta_4 \eta_5 \) and \( \tilde{h}_2 \eta_6 \). Finally, the result follows by expanding the exponential representation of \( \det(X + \tilde{h} \eta^T) \) as in Eq. (A.45) and using Eq. (A.95).

3. Conclusion

We have shown that the main results of Refs. [1,2] follow directly from an identity of general interest attributed to Capelli. Our result shows the usefulness of Capelli’s identity by putting the results of Refs. [1,2] in an unifying perspective. The development of computational procedures might be of interest in order to explore the usefulness of Capelli’s identity and further generalizations (see Ref. [3]). Also, it would be interesting to verify if expressions similar to Eq. (8) can be obtained from the other Cayley-type identities introduced in Ref. [3].

Note added in proof

After the completion of this work we became aware of some previous results related to Refs. [1,2]. See the recently published review article F.F. Abeles, Linear Algebra Appl. 454 (2014) 130-137 and references therein and K. Said, A. Salem, R. Belgacem, A mathematical proof of Dodgson’s algorithm, arXiv:0712.0362.

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References