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A note on a determinant identity

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ABSTRACT

In this note we show that the determinant identities obtained by Rezaifar and Rezaee (2007)–[1] and Dutta and Pal (2011)–[2] are straightforward consequences of a general result due to Capelli.

1. Introduction

In two interesting papers [1,2] the authors put forward an identity involving a representation for a determinant. Furthermore, they explore their approach in programming and establish a comparison with other standard existing methods (see, e.g., see section 7 of [1]). Here, our aim is to show that the results of Refs. [1,2] follow directly applying a well known general result due to Capelli (1855–1910) (see Ref. [3] and references therein or Eq. (1) below).

This note is organized as follows. In Section 2, we use the general formula of Capelli to recover the results of Refs. [1,2]. We follow closely Ref. [3]. See also Ref. [4]. In Section 3 we make some concluding remarks.

2. New proof of the determinant identity

Let us first give some simple definitions to fix notation. Let $X = (x_{ij})$ be an $n \times n$ matrix. We take $I, J \subseteq \{1, \ldots, n\}$, $I \cup I^c = \{1, \ldots, n\}$ and $I \cap I^c = \varnothing$. Also, we define $\mathcal{X} \equiv \det X$, $\partial_{IJ} = (\partial/\partial x_{ij}) = (\partial_{ij}) = (\partial_{i \in I j \in J})$, $X_{I^cJ^c} = (x_{ij}) = (x_{i \in I^c j \in J^c})$, $\mathcal{X}_{I^cJ^c} = \det X_{I^cJ^c}$ and $\epsilon(I, J) = (-1)^{\sum_{i \in I} i + \sum_{j \in J} j}$. The following identity is generally attributed to Capelli (see Eq. (1.2) in Ref. [3]):

$$\det(\partial_{IJ})\mathcal{X}^s = s(s+1)\ldots(s+k-1)\mathcal{X}^{s-1}\epsilon(I, J)\mathcal{X}_{I^cJ^c}, \tag{1}$$

with $|I| = k = |J|$. We refer the reader to Ref. [3] for further details towards the proof of this identity and a number of others generalizations, using methods from quantum field theory, like Grassmann–Berezin calculus. The identity in Eq. (1) is our starting point to show that

$$\mathcal{X} = \frac{1}{\mathcal{X}_{\{1,n\}^c\{1,n\}^c}} \det \begin{pmatrix} \mathcal{X}_{\{1\}^c\{1\}^c} & \mathcal{X}_{\{1\}^c\{n\}^c} \\ \mathcal{X}_{\{n\}^c\{1\}^c} & \mathcal{X}_{\{n\}^c\{n\}^c} \end{pmatrix}, \tag{2}$$

which is equivalent to Eq. (3.1) of Ref. [1] by a suitable adjustment of notation. Indeed, let us take Eq. (1) with $I = J = \{1, n\}$ (then $|I| = |J| = k = 2$) and $s = 2$. Therefore:

$$\det\left(\partial_{\{1,n\},\{1,n\}}\right)\mathcal{X}^2 = 6\mathcal{X}\epsilon(\{1,n\},\{1,n\})\mathcal{X}_{\{1,n\}^c\{1,n\}^c}, \tag{3}$$

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where we recall that

$$\det\left(\partial_{\{1,n\},\{1,n\}}\right) = \det\begin{pmatrix} \partial_{11} & \partial_{1n} \\ \partial_{n1} & \partial_{nn} \end{pmatrix}$$

and $\mathcal{X}_{\{i\}^c\{j\}^c}$, $\mathcal{X}_{\{1,n\}^c\{1,n\}^c}$ is the determinant of the matrix $(n-1)\times(n-1)$, $(n-2)\times(n-2)$ obtained from X by deleting line i and column j, the lines and columns 1 and n, respectively. Now observe that $\varepsilon(\{1,n\},\{1,n\}) = (-1)^{2(1+n)} = 1$, therefore we can write for Eq. (3)

$$\det(\partial_{\{1,n\},\{1,n\}})\mathcal{X}^2 = 6\mathcal{X}\mathcal{X}_{\{1,n\}^c\{1,n\}^c}. \tag{4}$$

A direct calculation gives

$$\det(\partial_{\{1,n\},\{1,n\}})\mathcal{X}^2 = 2\mathcal{X}\left[\det(\partial_{\{1,n\},\{1,n\}})\mathcal{X}\right] + 2\left(\partial_{11}\mathcal{X}\partial_{nn}\mathcal{X} - \partial_{1n}\mathcal{X}\partial_{n1}\mathcal{X}\right). \tag{5}$$

Now we apply Eq. (1) once again, this time with $I = J = \{1,n\}$ (then $|I| = |J| = k = 2$) and $s = 1$ to get

$$\det(\partial_{\{1,n\},\{1,n\}})\mathcal{X} = 2\epsilon(\{1,n\},\{1,n\})\mathcal{X}_{\{1,n\}^c\{1,n\}^c} = 2\mathcal{X}_{\{1,n\}^c\{1,n\}^c}. \tag{6}$$

Now, taking Eqs. (5), (6) and going back to Eq. (4), observe that

$$\mathcal{X}\mathcal{X}_{\{1,n\}^c\{1,n\}^c} = \partial_{11}\mathcal{X}\partial_{nn}\mathcal{X} - \partial_{1n}\mathcal{X}\partial_{n1}\mathcal{X}. \tag{7}$$

Using, conveniently, the standard Laplace expansion of the determinant $\mathcal{X}$, we can write

$$\mathcal{X} = -\sum_j (-1)^j x_{1j}\mathcal{X}_{\{1\}^c\{j\}^c} \Rightarrow \begin{cases} \partial_{11}\mathcal{X} = \mathcal{X}_{\{1\}^c\{1\}^c} \\ \partial_{1n}\mathcal{X} = (-1)^{1+n}\mathcal{X}_{\{1\}^c\{n\}^c}, \end{cases}$$

$$\mathcal{X} = \sum_j (-1)^{n+j} x_{nj}\mathcal{X}_{\{n\}^c\{j\}^c} \Rightarrow \begin{cases} \partial_{nn}\mathcal{X} = \mathcal{X}_{\{n\}^c\{n\}^c} \\ \partial_{n1}\mathcal{X} = (-1)^{1+n}\mathcal{X}_{\{n\}^c\{1\}^c}. \end{cases}$$

Using the results above in Eq. (7) we have

$$\begin{aligned} \mathcal{X}\mathcal{X}_{\{1,n\}^c\{1,n\}^c} &= \mathcal{X}_{\{1\}^c\{1\}^c}\mathcal{X}_{\{n\}^c\{n\}^c} - (-1)^{2(1+n)}\mathcal{X}_{\{1\}^c\{n\}^c}\mathcal{X}_{\{n\}^c\{1\}^c} \\ &= \mathcal{X}_{\{1\}^c\{1\}^c}\mathcal{X}_{\{n\}^c\{n\}^c} - \mathcal{X}_{\{1\}^c\{n\}^c}\mathcal{X}_{\{n\}^c\{1\}^c}. \end{aligned} \tag{8}$$

and we get Eq. (2) of this note or, equivalently, Eq. (3.1) of Ref. [1]. It is clear from the procedure outlined in this section that the main result stated in Ref. [2], more precisely, Eq. (1) there, is also a consequence of Eq. (1) by taking $I = \{i,j\} = J$ (note that $\varepsilon(\{i,j\},\{i,j\}) = 1$).

We close this section by noting that all the results stated here can be alternatively restated entirely in terms of the algebraic/combinatorial framework of Ref. [3]. We will limit ourselves to an indication of the main steps involved. Indeed, Eq. (8) follows from (4) and the following results of Ref. [3]. In what follows all the equations mentioned concern Ref. [3]. First, we take the Grassmann integral representation of $\det(\partial_{IJ})(\det X)^s$ in Eq. (5.13a) with $s = 2$ and $I = \{1,n\} = J$ and use Eq. (4.1) to obtain a Grassmann-type representation for $\det(X + \bar{\eta}\eta^T)$, following the notation of Ref. [3]. Next, we use the properties of the Grassmann integral (see Eq. (A.56)) and we observe that the product $(\prod \bar{\eta}\eta)_{I^c,J^c} = \bar{\eta}_2\eta_2\cdots\bar{\eta}_{n-1}\eta_{n-1}$ in the right-hand side of Eq. (5.13a) will select only certain elements of the matrix $\bar{\eta}\eta^T$ in the Grassmann (exponential) representation of $\det(X + \bar{\eta}\eta^T)$, i.e., $\bar{\eta}_1\eta_1$, $\bar{\eta}_1\eta_n$, $\bar{\eta}_n\eta_1$ and $\bar{\eta}_n\eta_n$. Finally, the result follows by expanding the exponential representation of $\det(X + \bar{\eta}\eta^T)$ as in Eq. (A.45) and using Eq. (A.95).

3. Conclusion

We have shown that the main results of Refs. [1,2] follow directly from an identity of general interest attributed to Capelli. Our result shows the usefulness of Capelli's identity by putting the results of Refs. [1,2] in an unifying perspective. The development of computational procedures might be of interest in order to explore the usefulness of Capelli's identity and further generalizations (see Ref. [3]). Also, it would be interesting to verify if expressions similar to Eq. (8) can be obtained from the other Cayley-type identities introduced in Ref. [3].

Note added in proof

After the completion of this work we became aware of some previous results related to Refs. [1,2]. See the recently published review article F.F. Abeles, Linear Algebra Appl. 454 (2014) 130-137 and references therein and K. Said, A. Salem, R. Belgacem, A mathematical proof of Dodgson's algorithm, arXiv:0712.0362.

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