The perception of a polyhedron in a generalized kaleidoscope: a perceptive experience

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Abstract: *This article focuses on the experience of viewing polyhedra in modified kaleidoscopes, highlighting the perceptive acts that take place. It presents aspects of geometry, such as concepts and ideas involved in the visualization of polyhedra in kaleidoscopes, through an analytical-mathematical study of the reflection of images in mirrors and generalized kaleidoscopes. In a phenomenological perspective, we call attention to the acts that are carried out in this experience and its possibilities of unfolding, leading the discussion to Merleau-Ponty ideas regarding the primacy of perception, which sustains the encounter of the being in the world.*

Introduction

In teaching situations, it is common for teachers to use didactic resources, such as geometric solids, to present spatial shapes to students. A polyhedron made of wood or other material can be observed, touched, moved to see what kind of noise it makes, to feel its surface, to understand what material it is made of. In these sensory experiences, different modes of feeling come into play. All of them are presentations that can be donated to the meaning of the polyhedron given in the experience. These are modes of perception of the object in which we also focus on the material properties of its representation and not just its geometric shape. All of them intertwine in the perception of the highlighted polyhedral object, but it is worth noting that, among the organs of sense, vision and touch allow the exploration of its geometric shape, thus giving us the polyhedron in presence.

We can also perceive the polyhedron by manipulating a polyhedral figure obtained by folding its planning. In this manipulation, at a given moment, only certain sides of the polyhedron are visible to our eyes, while the others, most of them perhaps, are absent, not visible. However, we intuit that there is something more than what is given to us by the organ of vision and we “see” more than what is in the visual plane in front of us. We know that at any moment we can rotate or walk around the polyhedron represented by the fold, and then the missing sides, faces and vertices will unveil in the field of our vision. When moving, our organ of vision does not focus on what was initially given, however, the perception of the object also brings the previous information, in a totality. Although the visual data at the beginning is absent from our field of vision, it is brought in the perceptive act when viewing other perspectives of the polyhedron. We can say that in the experience of the perception of polyhedron we see more than what is given to us by the image generated in our retina.

Thus, the act of perceiving, in a phenomenological approach, goes beyond what is given to us by the organs of sense: perceiving is not just feeling sensations, as when groping a rough surface, hearing a harmonious sound or touching a flat surface. “To perceive an object is to intend it and make it meaningful, through an intuition originally donor” (Ales Bello, 2006). The perception is a complex act different from a sensation, even though it is present in this act. Upon hearing a sound, we identify the melody, reactivate feelings like liking, hating, calming down. We put in perspective the sound that was given in perception. We perceive sound as a presence. But what is given in the act of perception is not the sound object itself.

As Merleau Ponty explains, perception is an act that gives us directly, without intermediaries, the one focused on the intentionality (Merleau-Ponty, 1999).

Below we present a study regarding the generation of images in kaleidoscopes and, afterwards, we seek to explain, in a merleau-pontiani phenomenological analysis, the perceptual acts that occur in the experience of visualizing polyhedra in generalized kaleidoscopes, bringing a synthesis of the discussion presented in Santos & Batistela (2019) which aims to advance in terms of understandings that can unfold in the direction of the geometric concepts involved.

Generating the image of a polyhedron in a generalized kaleidoscope

Initially, we present a brief explanation regarding the generation of images in generalized kaleidoscopes, highlighting concepts and mathematical properties that support the understanding of this physical-geometric process, indicating some broader studies on the subject.

The Polyhedras belong to the set of three-dimensional spatial objects. All faces on its surface are flat, the intersection of two faces is called the edge, and the intersection of three or more edges is called the vertex. When you have the same array of polygons distributed around each vertex and these polygons are regular and congruent, you have a regular polyhedron, classified as a Plato polyhedron (note that not all Plato's polyhedron is necessarily regular).

Archimedes' polyhedra are semi-regular polyhedra, that is, they have regular faces, but not all of them congruent with each other. To exemplify, consider an Archimedean rhombicosidodecahedron (figure 1, (5)). Its vertices have the following arrangement of polygons around them, arranged in a clockwise direction: triangle, square, pentagon and square. For this reason, it is denoted polyhedron (3,4,5,4), in reference to the number of sides of the faces. This Archimedes polyhedron can be obtained by truncating the platonic icosahedron (Figure 1): from the platonic icosahedron (1), the first truncation of its vertices is carried out and then the truncated Archimedean icosahedron (2) is obtained. With a second application of the truncation process, the Archimedean icosidodecahedron (3) is obtained, which is again submitted to the process generating a solid that has rectangles on its faces. When the rectangles are transformed into squares, the Archimedean truncated icosidodecahedron is obtained (4). Finally, truncating this latter solid, the Archimedean rhombicosidodecahedron is obtained (5).

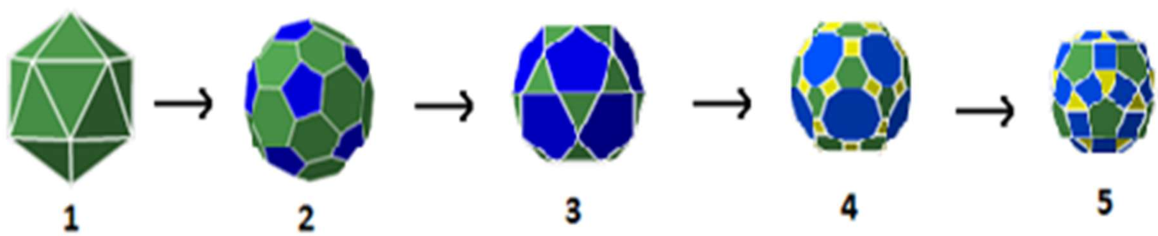

Figure 1: Generation process of the Archimedean rhombicosidodecahedron, or polyhedron (3,4,5,4

We can obtain some polyhedra through the generation of images in generalized kaleidoscopes together with regions built ad hoc on paper, which are placed inside it, called generating bases. Batistela (2005) presents a study of the geometric relationships involved in the reflection of images between mirrors that allows to understand, analytically, how they occur, highlighting the use of kaleidoscopes on getting Archimedean polyhedra by making bases obtained from the planning and cutting of a polygonal configuration. A generalized kaleidoscope is formed by three flat mirrors arranged in the form of a trihedral, which makes it possible to look at spatial figures. Such kaleidoscopes allow the generation of images with a three-dimensional aspect, which can generate polyhedra when the base is placed inside it (Figure 2).

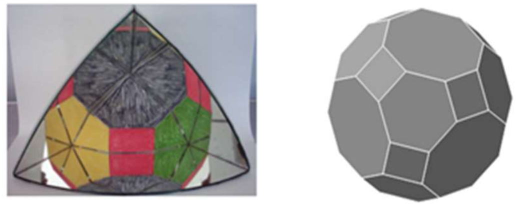

Figure 2: Generalized kaleidoscope and base for viewing the rhombitruncated cuboctahedron, or polyhedron (4, 6, 8).

The kaleidoscope allows the reflection of images in which we can see some faces, vertices and edges that give us the idea of the three-dimensional polyhedron.

The visualization of the polyhedron occurs because in any reflection in a flat mirror, the object and the image are equidistant from the plane of the mirror (BATISTELA, 2005). Then all the images of a point placed inside the kaleidoscope belong to the sphere with center at the point of intersection of the planes of the three mirrors. Such planes intersect the sphere determining a spherical triangle with angles 180 / l, 180 / m, 180 / n where l, men are divisors of 180. The result of the reflection of this spherical triangle in the mirrors is the division of the whole sphere in a network of triangles (Figure 3) that contain the image of any object placed inside the kaleidoscope.

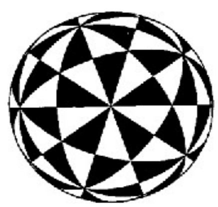

Figure 3: Spherical triangles covering a sphere

From the study of the determined spherical relationships, we can find the values of l, m and n that allow the perfect generation of spherical triangles that cover the whole sphere, without overlaps or gaps (BATISTELA, 2005). Ball & Coxeter (1987) show that the values of l, m and n that satisfy the condition are (2,2,3), (2,2,4), (2,2,5), (2,3,3), (2,3,4) and (2,3,5). Then, each suit corresponds the kaleidoscopes whose angles are the result of the division if 180º by this suit. For the (2,3,3), for example, corresponds to kaleidoscope with dihedral angles (90º, 60º, 60º). The central angles of the circular sectors formed by the mirrors can be deduced by the cosine law in spherical geometry. In the example considered, the kaleidoscope should have the following central angles: 70º52´, 54º54´ and 54º54´.

For the construction of the generalized kaleidoscope (Figure 4), it is suggested to truncate the circular sectors of each mirror (Figure 4. (a)), to facilitate handling (BATISTELA, 2005).

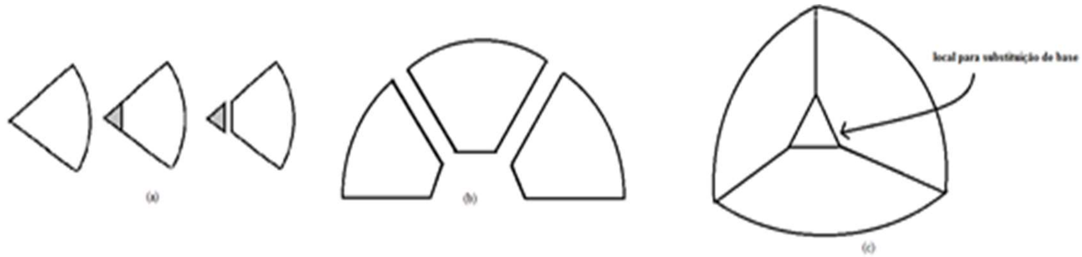

Figure 4: (a) mirror model in the form of circular sector and truncation of the circular sector; (b) planned kaleidoscope; (c) scheme of a generalized kaleidoscope and indication of the location for the placement of the base.

In Batistela (2005), we can follow the construction of the kaleidoscope and the respective base for the visualization of rhombicosidodecahedron (Figure 1 (5)). The generalized kaleidoscope has dihedral angles of (90º, 60º, 36º) and angles of the circular sectors (20º54´, 31º43´, 37º23´). The figure 5 shows the steps for making the base (made of paper) for visualization, when placed inside the kaleidoscope, of the Archimedean rhombicosidodecahedron. The space of 12º between the triangle and the square (Figure 5 (b)) should be cut out e, after properly folding, the base polygons are perpendicular to the mirrors.

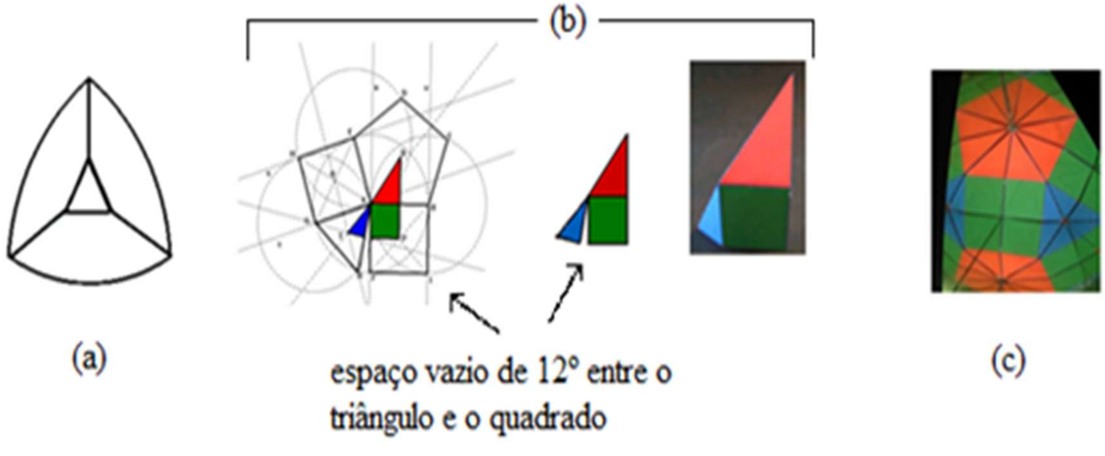

Figure 5: (a) generalized kaleidoscope (b) base constructed with ruler and compass, cut and folded (c) generation of the Archimedean rhombicosidodecahedron in the kaleidoscope.

In this explanation about the generation of images in kaleidoscopes, the importance of mathematical concepts and properties for understanding the image generation process between mirrors and for making of bases and the kaleidoscope. In the pedagogical scope, several studies (Murari, 1999, Martins, 2003; Almeida, 2003; Batistela, 2005; Reis, 2006; Gouveia, 2005; Batistela, 2006 and 2013; Buske, 2007; Neves, 2011) using this resource for teaching geometry, highlighting the esthetic appeal that mirrors and reflections exert, fascinating by the generation of images.

The perception of polyhedron in kaleidoscopes

The representation of a polyhedron in a wooden figure, for example, even made with great precision, cannot be considered the polyhedron itself, in terms of the ideality of that mathematical object. When we think of the representation given by the visualization in kaleidoscopes, it may seem even more distant, in principle, the relationship between the polyhedron and the image generated in the kaleidoscope. On the other hand, in the perceptual experience of visualizing in the kaleidoscope, this distance does not happen, and when we look at the polyhedral image generated in the kaleidoscope together with the base, we immediately identify the idea of the polyhedron.

Figure 6: visualization of the polyhedron (4,6,6) in a generalized kaleidoscope

Perceiving the polyhedron in the wooden figure or in the kaleidoscope image are different experiences, however, both experiences converge to the same idea: the idea of a polyhedron.

In the perceptual experience of the polyhedron generated in kaleidoscopes, at first sight, there is an intuition that highlights in our view the polyhedral object that appears in the reflection of images (Figure 6). The vision stands out in the perception of the geometric shape of the polyhedron. It is the organ of meaning that gives us the first intuition of the perceived polyhedral object generated between mirrors.

Merleau-Ponty (2004) highlights that what we achieve with our eyes is within our reach and is part of the motor project of our living-body:

> I just need to see something to know how to join it and reach it, even if I don't know how it happens in the nervous machine. My mobile body has the

> visible world, it is part of it, and that is why I can direct it in the visible. But it is also true that vision depends on movement. You only see what you look at. What would the vision be without any eye movement, and how would that movement not confuse things if he himself was reflex or blind, if he did not have his antenna, his clairvoyance, if the vision did not anticipate him? (MERLEAU-PONTY, 2004, p.14)

We cannot manipulate the image in the kaleidoscope without it coming undone, but we can, in visualization, infer analysis of properties and characteristics that stand out in the perceptual and its unfolding. By carefully observing the generated polyhedron, we can make statements such as how many sides, vertices and edges it has, what is the ratio between measures, which angles are congruent. The subject who visualizes can perceive the presence of the polyhedron as an object that donates itself in its entirety, given in the visuality of the images generated in the kaleidoscope that brings the idea of polyhedron, even though we can never see the "back" faces, because we cannot rotate the image or move around it without it disappear. But even though hidden faces can never be seen, they can be intuitive in the act of perceiving. The visualized polyhedron becomes present for the subject who intends it, being able to intuit relationships and properties, even though he cannot "see" the non-visible faces, as he could in the solid wood.

The reflected image is a mechanical effect of the play of light and colors between the mirrors, but if we recognize the object in it, it is the thought that creates this link, that "deciphers" the generated image, (re)signifying it (MERLEAU-PONTY, 2004). Thought, here, tells of a movement of meeting and articulation of meanings, which also brings memories, imagination, comparisons, logical organizations, etc.

Colors, depth, angle and orientation also guide this understanding. The vision is, as Merleau-Ponty (2004) tells us, stimulated to think with them. It is a see in action (Merleau-Ponty, 2004). Quality, light, color, direction that are there before us, there they are only because they awaken an echo in our living-body, because it welcomes them (Merleau-Ponty, 2004).

In this sense, Merleau-Ponty (2004, p. 27) opposes Descartes for whom the body is only the depository of the vision. He contradicts this understanding and claims that

> The space is counted from me as a zero point or degree of spatiality. I do not see it according to its outer shell, I live it inside, I am enclosed in it. Thinking well, the world is around me, not in front of me. (MERLEAU-PONTY, 2004, p. 27)

The philosopher also highlights: “the eye is that which was moved by a certain impact on the world” (MERLEAU-PONTY, 2004, p. 22). If I see the polyhedron in the kaleidoscope, I only see it through the elements that are there and that awaken the meaning of the polyhedron to the intentional look of my living-body. The act of perceiving polyhedron in visualization in kaleidoscopes emphasizes that the target object is the intentional correlate of a consciousness that aims at it. It is a meeting in which the gaze focuses precisely on this and not something else, in this way and not another. The perceived in the perception opens up to an understanding that extrapolates the individual parts of the object towards the articulations that take place in the actions of consciousness aiming at the totality of the intended object, in this case, the polyhedron.

The whole polyhedron can be perceived, even if it is not the kaleidoscope or the base. The unity that this set (kaleidoscope and base) gives us allows, in the perceptual act, to focus on the polyhedron, intending it. The visualization made possible offers the experience of perception of the object, although the reflection in the mirrors is not the polyhedron itself. In fact, the base and the kaleidoscope looked at separately do not bring any idea of a polyhedron and, perhaps, someone who had no contact with such objects could never imagine the relationship between them and the possibility of visualizing a polyhedron.

The perception made possible by the two objects, purposely constructed and articulated to generate predictable images between the mirrors, leads us to a new experience of perception of the polyhedron, differently from the experience with the solid or its folding. The composition formed by the base and the kaleidoscope gives us a “new” intentional object that, in principle, carries nothing from the two, individually considered, but that would not exist without them. In this visualization experience, we do not stop, initially, to think about how the mechanical process of reflection of images occurs, what is the angle between the mirrors or how the reflection of the base occurs. We only turn to the totality of the generated image.

When we focus on the polyhedron, we do not pay attention to the elements involved so that it could be visualized. On the other hand, we visualize much more than what the “structure” mounted between the mirrors gives us, since, under no circumstances, we could see the hidden faces of the polyhedron, even walking around it. But we do not question the possibility that these faces exist or “be there”, forming the totality of the polyhedron. Not even the polyhedron itself is put under suspicion: although it cannot be touched or manipulated, it is there, given in the intentional perception of the consciousness that focuses on it.

There is an encounter between intentional consciousness and the intended object.

> This encounter is the moment of perception that is not only subjective, since it is an intentional act of consciousness by which the thing seen is linked

> and, thus, given to consciousness, as a sense perceived in the act of perception or in experience. It is an immediate seeing, understood as an intuition of what is seen. It is a meaningful seeing, since what is seen is understood in its entirety, that is, in its figure, or core, and background, its surroundings, or context (BICUDO, 2010, p. 30).

The entire structure, including the kaleidoscope, the generating base and the way they are arranged, provides real and virtual images that complement each other, generating parts of the intended polyhedron, giving us the polyhedron as a presence. This structure donates itself as a tenuous backdrop that, at a later time, can be highlighted and investigated. We can ask ourselves: what angle with the mirror allows a side of the base and its image to have the necessary flatness to generate a side of the polyhedron? What is the relationship between the base measurements and the generated polyhedron?

Here we move towards other acts that aim to find out more about the visualized. In this case, the manipulation of the pieces, even if it results in the extinction of the polyhedron, will allow us to infer hypotheses and compare results to understand aspects of this structure that, now, becomes the main focus of intentionality. The polyhedron is not the object of prominence, or the core, but it continues to donate itself as a background that makes it possible to make inferences and conclusions and advance the investigations.

When focusing on the visualization generated in the mirrors, we observe that the base contains only parts of the faces of the polygons that form the polyhedron and these parts are mathematically planned so that its reflection in the mirrors generates a complete face of this polyhedron. In this way, the faces of the polyhedron are made up of a "real" part and other virtual parts that are generated by the reflection of the first in the mirrors. When the images bounce between the mirrors, the real and virtual parts merge, eliminating the split between them, allowing the visualization of a complete face of the generated polyhedron. New reflections give us other faces that form the polyhedron.

As we can see, the observer's attentive gaze can lead to observations and analyzes that require reasoning and verification which, in turn, allow us to understand the polyhedron's constitution. Batistela (2005), in his study on the construction of the bases for generalized kaleidoscopes for the visualization of Archimedes' polyhedra, describes that the process of discovering the bases "occurred by trial and error" (Batistela, 2005) when trying to locate in the polyhedron some configuration that could fit the angles of the kaleidoscopes. For example, for the case of the polyhedron (4, 6, 6) visualized in the generalized kaleidoscope with dihedral angles (90º, 60º, 60º) (Figure 6), the base was found looking for in the polyhedron planning a polygonal region that could be reflected in the mirrors originating the faces of this polyhedron (Figure 7) (BATISTELA, 2005).

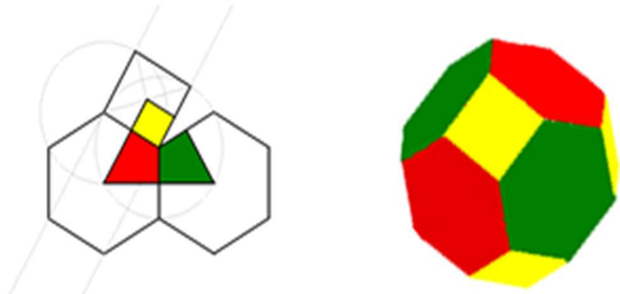

Figure 7: Kaleidoscopic base that allows the visualization of the polyhedron (4,6,6), elaborated from the planning of the rhombitruncated cuboctahedron

The constitution of the polyhedron and the actions triggered, or that can be triggered, have in common perceptions made possible by the intentional look of the subjects involved. Advancing in the direction of geometric understandings requires developments when we return to the perceptive acts and cognitive processes, among which are recall, imagination, comparison, and turning on these acts in reflexive actions. It also requests that the perceived can be expressed and shared with the co-subjects, being able to advance through understandings among those involved, making intersubjective construction possible.

Considerations

Paying attention to the classroom or teaching situations, the exploration of the resources discussed in this article, such as kaleidoscopes, bases, solids, folds, with a focus on intentional acts and their expressions may indicate possible ways for developments and learning to occur in exchange understanding between students and in the search for new re-elaborations, including towards the cognitive chains necessary for the analytical-geometric and formal study of the geometric ideas involved.

Geometry science, in its constitution, needs articulations that direct the experiences lived, understood and expressed by language, allowing new articulations, providing a latent field of knowledge in its possibilities.

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